(丙組)

一.Compute the following problems:

(1) $\int \ln(1-\sqrt{x})\,dx$ (5%)

(2) Let $F(x) = \int_{x-1}^{x+1} (\sin t) \exp(-xt^2)\,dt$. Find $F'(0)$=? (5%)

(3) $\lim_{x\to\infty} x^2\sin(1/x)$ (5%)

二.Let $f(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$

(1) Find the domain of the funtion f. (5%)
(i.e. Find the interval of convergence of the power series)

(2) Find the sum of the power series. (5%)

(3) Find $f'(x)$=? (5%)

三. Let $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(1) Find $D_1f(0,0)$ (5%)

(2) $D_{12}f(0,0)$ (5%)

(3) Show that $D_{21}f(0,0) \neq D_{12}f(0,0)$ (5%)

四.(1) Show that $\ln(x) \leq x-1$, for all $x \geq 0$. (5%)

(2) Suppose $\sum_{i=1}^{m} P_i = 1$, $\sum_{i=1}^{m} Q_i = 1$ and $P_i > 0$, $Q_i > 0$ for $i=1,\ldots,m$.

Using (1), Show that $\sum_{i=1}^{m} P_i \ln(P_i/Q_i) \geq 0$. (5%)

五.Let f be a nonnegative continuous function in [a,b].

(1) If $\int_a^b f(x)dx=0$, show that $f(x)=0$ for all $x \in [a,b]$. (10%)

(2) If M denote the maximum value of f on [a,b],
show that $\lim_{n\to\infty} \left[\int_a^b (f(x))^n dx \right]^{1/n} = M$. (10%)

六.Define $f(x) = \left[\int_0^x \exp(-t^2)dt \right]^2$, $g(x) = \int_0^1 [\exp(-x^2(t^2+1))]/(t^2+1)\,dt$.

(1) Show that $g'(x) + f'(x) = 0$ for all x . (10%)

(2) Use (1) to show that $g(x) + f(x) = \pi/4$. (8%)

(3) Use (2) to prove that $\int_0^\infty \exp(-t^2)\,dt = \frac{1}{2}\sqrt{\pi}$. (7%)