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1. A urn contains 7 balls, $\theta$ of which are red. A sample of size 2 is drawn without replacement to test $H_0 : \theta \leq 1$ against $H_a : \theta > 1$. If the null hypothesis is rejected if one or more red balls are drawn, find the power function of the test. (10%)

2. Let $X$ and $Y$ be discrete random variables whose joint probability distribution with parameter $\theta$ is defined by this table: (15%)

| | | $X$ = 1 | $X$ = 2 |
|---|---|---|---|
| $Y$ | 1 | $\frac{1}{4} - \theta$ | $\frac{1}{4} + \theta$ |
| | 2 | $\frac{1}{4} + \theta$ | $\frac{1}{4} - \theta$ |

   (a) Show that $S = |X - Y|$ is a sufficient statistic for $\theta$. (5%)

   (b) Are $S$ and $X$ independent? (10%)

3. Suppose $X_1, \ldots, X_n$ are i.i.d. from Poisson($\lambda$). Let $Y$ be the number of $X_i$'s that takes the value 0. (15%)

   (a) Show that $Y$ follows the binomial distribution with parameters $n$ and $p$. Find an expression for $p$ as a function of $\lambda$. (5%)

   (b) Using the method of moments, show that an estimator, $\tilde{\lambda}$, of $\lambda$ based on $Y$. (5%)

   (c) Find the asymptotic distribution of $\sqrt{n}\left(\frac{Y}{n} - p\right)$. (5%)

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4. A two-stage experiment was performed on a biased coin which had probability $p$ of coming up heads and $1-p$ of coming up tails, where $0 < p < 1$. The first stage of the experiment consisted of tossing this coin a known total of $m$ times and recording $X$, the number of heads. At the second stage, the coin was tossed until a total of $X+1$ tails had come up. The number $Y$ of heads observed at the second stage along the way to getting the $X+1$ tails was then recorded. The experiment was repeated a total of $n$ times and the counts $(X_i, Y_i)$ for each experiment, $i = 1, \ldots, n$, were recorded. (20%)

   (a) Find the maximum likelihood estimator of $p$. (5%)

   (b) Find the sufficient statistic, $T$, of $p$. (5%)

   (c) Find $E(T)$ and $\mathrm{Var}(T)$. (10%)

5. Let $X_1, \ldots, X_n$ be a random sample with uniform distribution on $[0, \theta]$ where $\theta > 0$ is unknown and let $X_{(1)}, \ldots, X_{(n)}$ be the corresponding order statistics. (40%)

   (a) Show that $X_{(n)}$ is a complete sufficient statistic. (10%)

   (b) Find $C$ such that $E(CX_{(1)}) = \theta$. (10%)

   (c) Find the conditional density of $X_{(1)}$ given $X_{(n)}$. (10%)

   (d) Compute $E(CX_{(1)}|X_{(n)})$ where $C$ is the same as (c). (10%)