國立成功大學 104 學年度碩士班招生考試試題

系所組別：統計學系
考試科目：數理統計
考試日期：0212，節次：2

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

(1) (10%) Let a random variable X have the probability density function (pdf):

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Set $Y_n = \cos(X/n)$. Show that Y_n converges in probability and determine the limit as $n \to \infty$.

(2) Let $X = (X_1, \ldots, X_n)$ be a random sample from a distribution with pdf given by

$$f(x|\theta) = \frac{cx^{c-1}}{\theta^c} e^{-(x/\theta)^c} I(x > 0),$$

where $c > 0$ is known.

(a) (5%) Find the uniformly minimum variance unbiased estimator (UMVUE) for θ.

(b) (10%) Find the uniformly most powerful (UMP) test of size α for testing

$$\mathrm{H}_0: \theta \le \theta_0 \quad \text{versus} \quad \mathrm{H}_1: \theta > \theta_0,$$

where θ_0 is a positive constant.

(3) (10%) Suppose we have a sample of size n from a distribution with the cumulative distribution function (cdf) given by

$$F(x|\alpha, \beta) = \frac{x}{\beta} I(0 \le x < \beta) + I(x > \beta), \quad \alpha > 0, \beta > 0.$$

Find the MLE's of α and β, respectively.

(4) Suppose $\boldsymbol{X} = (X_1, X_2)'$ has a bivariate normal distribution with mean vector μ and covariance matrix, $\Sigma = (1-\rho)I_2 + \rho J_2$, where I_2 is an identity matrix of order 2, and J_2 is a 2×2 matrix of 1's. Let $Q_1 = (X_1 - X_2)^2$ and $Q_2 = (X_1 + X_2)^2$.

(a) (5%) Derive the range of ρ.

(b) (5%) Find the distributions of Q_1 and Q_2, respectively.

(c) (5%) Are Q_1 and Q_2 independent? Justify your answer.

(5) Suppose $X_1, X_2, \ldots, X_n$ is a random sample having one parameter Topp-Leone distribution whose pdf is given by

$$f(x) = \theta(2-2x)(2x-x^2)^{\theta-1}, \quad 0 < x < 1, \ \theta > 0,$$

where θ is the shape parameter and we write $X_i \sim TL(\theta)$.

(a) (5%) Find the cdf of X.

(b) (10%) A prior for the parameter θ is assumed to be

$$\pi(\theta) \propto \frac{1}{\theta}, \ \theta > 0.$$

Find the Bayes estimator and risk of θ under squared error loss function (SELF).

(c) (5%) Suppose $X \sim TL(\theta_1)$ and $Y \sim TL(\theta_2)$ and X and Y are independent. We define the stress-strength parameter as $\delta = P(X > Y)$. Please express δ in terms of θ_1 and θ_2.

(6) Suppose that $X_1, \ldots, X_n$ is an independent and identically distributed (iid) sample with size n from the Poisson distribution with mean λ. We are interested in estimating $\theta = P(X_1 = 0) = e^{-\lambda}$. Consider the following two estimators:

$$T_n^1 = e^{-\overline{X}_n}, \quad T_n^2 = \frac{1}{n}\sum_{i=1}^{n} I\{X_i = 0\},$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $I\{\cdot\}$ is the indicator function.

(a) (5%) Find the asymptotic distribution of T_n^1.

(b) (5%) Find the asymptotic distribution of T_n^2.

(c) (5%) Which estimator is more efficient in estimating θ when a large sample size is available? Show your argument.

(7) Let $X_1, \ldots, X_n$ be a sample from probability mass function

$$P(X = k) = \begin{cases} \frac{1}{N}, & k = 1, 2, \ldots, N, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (5%) Find the maximum likelihood estimator $\hat{N}$ of N.

(b) (5%) Show that $P(\hat{N} > k) = 1 - \left(\frac{k}{N}\right)^n$ for $k = 1, 2, \ldots, N$.

(c) (5%) For sample size $n = 2$, compute $E[\hat{N}]$.