※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

(1) Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution on (a, ∞) with scale parameter θ, where $\theta > 0$ and $a \in R$.

(a) (5%) Find the UMVUE of a when θ is known.

(b) (5%) Find the UMVUE of θ when a is known.

(c) (5%) Find the UMVUE of θ and a.

(d) (5%) Find the UMVUE of $P(X_1 \geq t)$ for a fixed $t > a$.

(2) Let X be an observation from the distribution with the probability density function given by

$$f(x|\theta) = \frac{c}{2}e^{\theta x - |x|}, \quad -1 < \theta < 1.$$

(a) (5%) Find the constant c in terms of θ.

(b) (5%) Show that if $0 \leq \alpha \leq 0.5$, then $\alpha X + \beta$ is admissible for estimating $E(X)$ under the squared error loss.

(3) Suppose that X has the discrete probability mass function given by

$$p(x|\theta) = \begin{cases} \frac{\theta}{2}, & \text{if } x = -1, 1; \\ 1-\theta, & x = 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) (2%) What is the MLE of θ?

(b) (3%) Find the UMVUE of θ.

(4) (5%) Let Y be a random variable and m be a median of Y. Show that, for any real numbers a and b such that $m \leq a \leq b$ or $m \geq a \geq b$, $E|Y-a| \leq E|Y-b|$.

(5) Suppose that X_1 and X_2 are two independent random variables from the uniform distribution $U(-1,1)$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

(a) (5%) Find the joint probability density function of Y_1 and Y_2.

(b) (5%) Find the marginal probability density function of Y_1.

(c) (5%) Find the mean and variance of Y_1 given $Y_2 = 0$ if they exist.

(6) Let $X_1, \ldots, X_n$ be a random sample from a distribution with the probability density function as

$$f(x|\theta) = \begin{cases} \theta(1-x)^{\theta-1}, & \text{if } 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) (2%) Find the form the uniformly most powerful test of $H_0 : \theta = 1$ against $H_1 : \theta > 1$.

(b) (3%) Find the likelihood ratio test for testing $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$.

(7) Let $Y = (X_1, X_2)'$ have a bivariate normal distribution with mean μ and covariance matrix Σ given by

$$\mu = \begin{pmatrix} 5 \\ 10 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 5\rho \\ 5\rho & 25 \end{pmatrix}.$$

Suppose that $\rho > 0$ and $P(6 < X_2 < 14 | X_1 = 5) = 0.68$. Note that $P(Z < 1) = 0.84$ when $Z \sim N(0, 1)$.

(a) (5%) Please determine the value of ρ.

(b) (5%) Let $U = X_1 + X_2$ and $V = X_1 - X_2$. Please determine the distribution of random vector (U, V). Are U and V independent? Explain.

(c) (5%) Find a matrix A and a real vector b such that

$$T = AX + b$$

follows a standard bivariate normal distribution with mean μ_T and covariance Σ_T given by

$$\mu_T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(d) (5%) Let $W = (Y - \mu)'\Sigma^{-1}(Y - \mu)$. Find the mean and variance of W^2.

(8) Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on the interval [0, 1] and let $R = X_{(n)} - X_{(1)}$, where $X_{(i)}$ is the ith order statistic.

(a) (5%) Derive the probability density function of R.

(b) (5%) Find the limiting distribution of $2n(1 - R)$.

(9) Let $\{T_n\}, \{W_n\}, \{X_n\}, \{Y_n\}, \{Z_n\}$ be sequences of random variables. Assume $\{T_n\}$ is bounded in probability, $W_n \xrightarrow{D} W$, $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{D} Y$, and $Z_n \xrightarrow{P} 2$. We denote $o_p(X_n)$ as

$$Y_n = o_p(X_n) \quad \text{if and only if} \quad \frac{Y_n}{X_n} \xrightarrow{P} 0, \text{ as } n \to \infty.$$

Decide whether the following statement is **True(T)** or **False(F)**.

(請勿在此作答)

(a) (1%) (　　　): $X_n \xrightarrow{D} X$.

(b) (1%) (　　　): $X_n Y_n Z_n \xrightarrow{D} 2XY$.

(c) (1%) (　　　): $\{X_n\}$ is bounded in probability.

(d) (1%) (　　　): $\{T_n\}$ converges in distribution.

(e) (1%) (　　　): $\frac{1}{Z_n} \xrightarrow{P} \frac{1}{2}$.

(f) (1%) (　　　): $W_n Y_n \xrightarrow{D} WY$.

(g) (1%) (　　　): $\sqrt{X_n} \xrightarrow{D} \sqrt{X}$.

(h) (1%) (　　　): $Y_n^2 \xrightarrow{P} Y^2$.

(i) (1%) (　　　): $o_p(T_n) \xrightarrow{P} 0$.

(j) (1%) (　　　): $Z_n o_p(T_n) \xrightarrow{P} 0$.