※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

一、True or False $(10 \times 2\% = 20\%)$

(For the following statements, please answer **T** if it is true and **F** otherwise.)

1. Both of random variables X and Y are normally distributed, then (X, Y) follows a bivariate normal distribution.
2. X is a continuous random variable, and $P(X \leq s + t | X \leq t) = P(X \leq s)$, then X follows an exponential distribution.
3. X and Y are i.i.d. standard normal random variables, then $E(X/Y) = 0$.
4. As in 3., $E(X^2/Y^2)=0$
5. If an estimator is consistent, then it is unbiased and its variance converges to zero as the sample size *n* goes to infinity.
6. Two disjoint events are independent.
7. The summation of n independent exponential random variables is an exponential random variable.
8. If $(X_1, X_2)'$ is jointly distributed as a bivariate normal distribution, then the conditional distribution of $X_1|X_2$ has to be a normal one as well.
9. If a random variable X is nonnegative, then $E(X) \geq 0$.
10. Under a symmetric distribution, the sample mean and sample variance are asymptotically independent.

二、Multiple Choice $(7 \times 5\% = 35\%)$

1. Which of the following statement(s) is/are true?

   i. For any population model, there is a sufficient statistics.

   ii. If a complete sufficient statistics exists, then it has to be minimal sufficient.

   iii. For any population model, there is a minimal sufficient statistics.

   (A) i. only (B) ii only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

2. Which of the following statement(s) is/are true?

   i. $X_n \xrightarrow{p} a,\ Y_n \xrightarrow{p} b$, then $X_n Y_n \xrightarrow{p} ab$

   ii. $X_n \xrightarrow{p} a,\ Y_n \xrightarrow{d} Y$, then $X_n Y_n \xrightarrow{d} aY$

   iii. $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} Y$, then $X_n Y_n \xrightarrow{d} XY$

   (A) i. only (B) ii only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

3. $(X,Y)'$ is uniformly distributed on $S_{(X,Y)} = \{(x,y); 0 \le x \le 1, 0 \le y \le 1, x+y \le 1\}$, then
   i. The marginal distribution of X is a uniform distribution.
   ii. $X|Y=y$ is a uniform distribution.
   iii. $P(X^2+Y^2 \le 0.5|X<Y) = \pi/4$

   (A) i. only (B) ii only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

4. $\boldsymbol{X}_n = (X_{1n}, X_{2n})'$ is a sequence of random vectors, which of the following statement(s) is(are) correct?

   i. If $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X} = (X_1, X_2)$, then $X_{1n} \xrightarrow{d} X_1$, $X_{2n} \xrightarrow{d} X_2$.

   ii. If $\boldsymbol{X}_n \xrightarrow{p} \boldsymbol{a} = (a_1, a_2)$, then $X_{1n} \xrightarrow{p} a_1$, $X_{2n} \xrightarrow{p} a_2$.

   iii. If $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X} = (X_1, X_2)$, and $Y_n = X_{1n} + X_{2n}$, $Y = X_1 + X_2$, then $Y_n \xrightarrow{d} Y$.

   (A) i. only (B) ii only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

5. Which of the following statement(s) is(are) correct?
   i. If the maximum likelihood estimator is unbiased, then it is the best unbiased estimator.
   ii. The most powerful test is a test with the lowest type I error rate under the same size α.
   iii. Let $K_\Phi(\theta)$ be the power function of the testing procedure Φ, then the type II error rate of Φ is the value of $K_\Phi(\theta)$ when θ is true.

   (A) i. only (B) ii only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

6. Let $X_1, X_2, \ldots, X_n, n > 1$ be the observations from a normal population with mean μ and variance σ^2, further, let $\bar{X}$ and S^2 be the sample mean and sample variance, respectively, then
   i. $\bar{X}$ is the UMVUE of μ.
   ii. S^2 is the UMVUE of σ^2.
   iii. $\bar{X}S^2$ is the UMVUE of $\mu\sigma^2$.

   (A) i. only (B) ii only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

7. Which of the following statement(s) is/are correct?
   i. If the UMVUE exists, then it is a function of the sufficient statistics.
   ii. The independence between events A and B cannot imply the independence between A^c and B^c.
   iii. For a fixed sample size n, The UMVUE is an efficient estimator.

   (A) i. only (B) ii only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

三、Problems (45%)

1. (15%) $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two k-dimensional random vectors such that $\boldsymbol{X}|\boldsymbol{Y} \sim N_k(\boldsymbol{Y}, \boldsymbol{\Sigma})$, and $\boldsymbol{Y} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Lambda})$, $\boldsymbol{\mu}$ is a k-dimensional constant vector and both of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda}$ are $k \times k$ covariance matrices. What is the marginal distribution of $\boldsymbol{X}$?

2. (15%) Prove the subadditivity of the probability function P($\cdot$) defined on $(\Omega, \mathcal{F})$, that is, let $A_i, i = 1, \ldots$ be the subsets of Ω, then

$$\mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathrm{P}(A_i)$$

3. (15%) Usually the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ of the random sample $X_1, X_2, \ldots X_n$ is a reasonable and good estimator for the population mean. Nevertheless, the sample mean can perform poorly from time to time. For example, the sample mean under the model of a uniform distribution $\mathrm{Uni}(0, \theta), \theta > 0$ is not such a good choice to estimate the population mean. Please find an unbiased estimator under this uniformly distributed model which is better than $\bar{X}$