國立成功大學 108 學年度碩士班招生考試試題

系　　所：統計學系

考試科目：數學

考試日期：0224，節次：1

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10%) Evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-(x-y)^2}}{1+(x+y)^2}\,dx\,dy.$$

2. (10%) For what values of x does the function

$$f(x) = 3 + |x-1| + |x+1|$$

have a unique inverse and find the corresponding inverse function?

3 (10%) Suppose that the function $f(x)$ is such that $f'(x)$ and $f''(x)$ are continuous in a neighborhood of the origin and satisfies $f(0)=0$. Show that

$$\lim_{x\to 0}\frac{d}{dx}\left[\frac{f(x)}{x}\right] = \frac{1}{2}f''(0).$$

4. (10%) Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the following condition: There is an r, $0<r<1$, such that

$$|a_{n+1}-a_n| < br^n, \qquad n = 1, 2, \ldots,$$

where b is a positive constant. Show that this sequence converges.

5. (10%) Show that $\lim_{n\to\infty}\int_0^{n\pi}\left|\frac{\sin x}{x}\right|dx = \infty$, where n is a positive integer.

6. (10%) Let A and B be $n\times n$ idempotent matrices. Show that $A-B$ is idempotent if and only if $AB = BA = B$.

7. Let A and B be $m\times n$ matrices. Show that
   a) (5%) $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$.
   b) (5%) $\text{rank}(A) = \text{rank}(A') = \text{rank}(A'A) = \text{rank}(AA')$.

8. (10%) If P is symmetric, then P is idempotent and of rank r if and only if it has r eigenvalues equal to unity and $n-r$ eigenvalues equal to zero.

9. (10%) Suppose A and B are nonsingular matrices, with A being $m\times m$ and B being $n\times n$. For any $m\times n$ matrix C and any $n\times m$ matric D, it follows that if $A+CBD$ is nonsingular then

$$(A+CBD)^{-1} = A^{-1} - A^{-1}C(B^{-1}+DA^{-1}C)^{-1}DA^{-1}.$$

10. Let A be $m\times m$ a nonsingular matrix and I $m\times m$ be an identity matrix such that $I+A$ is nonsingular, and define

$$B = (I+A)^{-1} + (I+A^{-1})^{-1}.$$

   a) (5%) Show that if x is an eigenvector of A corresponding to the eigenvalue λ, then x is an eigenvector of B corresponding to the eigenvalue 1.
   b) (5%) Show that $B = I$.