國立成功大學 109 學年度碩士班招生考試試題

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Consider the $m \times m$ matrix $A = \alpha \mathbf{I}_m + \beta \mathbf{1}_m \mathbf{1}_m'$, where α and β are scalars, $\mathbf{I}_m$ is the $m \times m$ identity matrix, and $\mathbf{1}_m$ is the $m \times 1$ vector having each component equal to 1.
   a) (5%) Find the eigenvalues and eigenvectors of A.
   b) (5%) Determine the eigenspaces and associated eigenprojections of A.
   c) (10%) For which values of α and β will A be nonsigular?
   d) (10%) Using a), show that when A is nonsigular, then

   $$A^{-1} = \alpha^{-1}\mathbf{I}_m - \frac{\beta}{\alpha(\alpha+m\beta)}\mathbf{1}_m\mathbf{1}_m'.$$

   e) (10%) Find the determinant of A.

2. Let matrices A, B, and C be given by

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & 2 \\ -1 & 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 5 & 3 \\ -2 & -1 & 1 \end{bmatrix}$$

   a) (5%) Which of these matrices are diagonalizable?
   b) (5%) Which of these matrices have their rank equal to the number of nonzero eigenvalues?

3. (10%) Evaluate the integral $\iint_\Omega \sin\left(\frac{y-x}{y+x}\right) dxdy$; Ω the region in the first quadrant bounded by the lines $x+y=1$ and $x+y=2$.

4. a) (10%) If f and $\partial f/\partial x$ are continuous, then show the function

$$H(t) = \int_a^b \frac{\partial f}{\partial x}(t,y)\,dy$$

is continuous.

b) (10%) Use the identity

$$\int_0^x \int_a^b \frac{\partial f}{\partial x}(t,y)\,dydt = \int_a^b \int_0^x \frac{\partial f}{\partial x}(t,y)\,dtdy$$

to verify that

$$\frac{d}{dx}\left[\int_a^b f(x,y)\,dy\right] = \int_a^b \frac{\partial f}{\partial x}(x,y)\,dy.$$

5. (10%) Find the points on the sphere $x^2+y^2+z^2=1$ that are closest to and farthest from the point $(2,1,2)$.

6. Set $f(x)=xe^x$.

a) (5%) Expand $f(x)$ in a power series.

b) (5%) Integrate the series in a) and show that

$$\sum_{n=1}^{\infty}\frac{1}{n!\,(n+2)}=\frac{1}{2}.$$