國立成功大學
114學年度碩士班招生考試試題

編　號：165

系　所：統計學系

科　目：數學

日　期：0211

節　次：第 1 節

注　意：1.不可使用計算機
2.請於答案卷(卡)作答，於試題上作答，不予計分。

1. (25%) Find the answers

(1) (5%) $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$

(2) (10%) $\int_0^3 \int_{\sqrt{1+x}}^2 \cos\left(\frac{x}{y+1}\right) dy\, dx$

(3) (10%) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x-y} (x^2+y^2)\, dz\, dy\, dx$

2. (10%) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n!\,(n+2)} = \frac{1}{2}$$

3. (10%) Consider the region in the xy-plane bounded by the curve

$$y = \frac{\sin x}{3+\cos^2 x},$$

the x-axis, and the vertical lines $x=0$ and $x=\pi$. This region is revolved about the y-axis to form a solid. Suppose the volume V of this solid can be expressed in the form

$$V = \frac{\pi^b}{a}$$

where a and b are real constants. Find (a, b).

4. (10%) Compute the surface area S of the solid formed by revolving the graph of

$$y = e^x, \qquad 0 \le x \le 1,$$

around the x-axis.

5. (10%) Let Q be an $m \times n$ real matrix with $m > n$ and $Q^T Q = I_n$, where I_n is the $n \times n$ identity matrix. Define $P = QQ^T$.

(1) (5%) Find $\det(P)$.

(2) (5%) Determine all the eigenvalues of the P, listing each with its multiplicity.

(Please detail your calculations)

6. (10%) Suppose A is an $n \times n$ complex matrix that is diagonalizable, and all of its eigenvalues lie in $\{0,1\}$. Find the smallest integer $k \geq 2$ such that $A^k = A$ holds for every such matrix A. Give a detailed explanation of how you arrive at your answer.

7. (25%) Let P_n be the $n \times n$ Pascal matrix whose entries are given by

$$P_{ij} = \binom{i+j-2}{i-1} \text{ for } i,j \in \{1,2,\dots,n\}.$$

(1) (5%) Let $n = 3$. Fin the eigenvalues of the Pascal matrix P_3.

(2) (10%) Prove that P_n admits an LU-decomposition $P_n = L_n U_n$. Specifically, let L_n (lower triangular) and U_n (upper triangular) be defined by

$$L_{ij} = \begin{cases} \binom{i-1}{j-1} & i \geq j \\ 0 & i < j \end{cases}, \quad U_{ij} = \begin{cases} \binom{j-1}{i-1} & j \geq i \\ 0 & j < i \end{cases}$$

Show that $P_n = L_n U_n$ by verifying these products reproduce the entries of P_n.

(3) (10%) Suppose n is even. Prove that the eigenvalues of P_n occur in reciprocal pairs (i.e., if λ is an eigenvalue, then so is $1/\lambda$).