Part of Calculas

I. **TRUE** or **FALSE** : Please explain your reasons.
Let f(x) be a real-valued function.
(1) If $f'(c) = 0$ then f(x) has a relative extremum at c. (8 pts.)
(2) If f(x) is continuous at c then f(x) is differentiable at c. (8 pts.)

II. Can you apply L'Hospital Rule to find the following limit ?
Please explain your reasons.

(1) $\lim\limits_{x \to \infty} \frac{x - \sin(x)}{x}$ (5 pts.)

(2) $\lim\limits_{x \to \pi/2} \frac{\cos(\sqrt{x})}{\cos(x)}$ (5 pts.)

III. Does the function $f(x) = \cos(\frac{1}{x^2})$ have the limit at 0 ? Why ? (6 pts.)

IV. Find the double integral of

$$\iint_R x^2y^2 \, dx\,dy,$$

where R is the region bounded by hyperbolas $xy=1$, $xy=2$ and lines $y=x$ and $y=2x$. (9 pts.)

V. Find the double integral of

$$\iint_R x^2 + y^2 \, dx\,dy,$$

where $R = \{ (x,y) : (x-2)^2 + y^2 \le 4 \text{ and } y \ge 0 \}$. (9 pts.)

Part of Linear Algebra

I. (12 points) Consider the symmetric matrix

$$A = \begin{bmatrix} 13 & -4 & 2 \\ \cdot & 13 & -2 \\ \cdot & \cdot & 10 \end{bmatrix}.$$

The eigenvalue-eigenvector pairs (λ_i, e_i), $i = 1, 2, 3$, of A are as the following

$$\lambda_1 = 9 \qquad e_1 = [1/\sqrt{2}, 1/\sqrt{2}, 0]^T,$$
$$\lambda_2 = 9 \qquad e_2 = [1/\sqrt{18}, -1/\sqrt{18}, -4/\sqrt{18}]^T,$$
$$\lambda_3 = 18 \qquad e_3 = [2/3, -2/3, 1/3]^T.$$

Let

$$A_1 = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(1). Find the determinant of A.
(2). Find the determinant of $A_1A_2A_3A_4A$.
(3). Is there a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_1E_2 \ldots E_kA = I$? Why?

II. (38 points) Suppose we collect two observation vectors $\mathbf{x}_1 = [x_{11}, x_{21}, \ldots, x_{n1}]^T$, $\mathbf{x}_2 = [x_{12}, x_{22}, \ldots, x_{n2}]^T$. Let $\mathbf{1} = [1,1,\ldots,1]^T$ be an $n \times 1$ vector.

(1). Find the projection matrix **D** for the space spanned by **1** and find the vectors of projection of $\mathbf{x}_1$ and $\mathbf{x}_2$ on sp(**1**), the subspace spanned by **1**.
(2). A matrix Q is said to be an idempotent matrix if and only if $Q^2 = Q$. Define $\mathbf{P} = \mathbf{I} - \mathbf{D}$, where **I** is an $n \times n$ identity matrix. Show that **D** and **P** are idempotent matrices. (5 points)
(3). Find all the eigenvalues of D and P. What is(are) the eigenvector(s) corresponding to the positive eigenvalue(s) of D? (6 points)
(4). Show that **P** is a projection matrix.
(5). Find the projection vectors of $\mathbf{x}_1$ and $\mathbf{x}_2$ on the space spanned by the column vectors of **P**. Let the two vectors of projection be $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Find the lengths of $\mathbf{e}_1$ and $\mathbf{e}_2$.
(6). If the angle formed by $\mathbf{e}_1$ and $\mathbf{e}_2$ is 30°, find the sample correlation coefficient of the two observation vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
(7). Let S be the sample variance-covariance matrix of $\mathbf{x}_1$ and $\mathbf{x}_2$. That is

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \quad \text{(7 points)}$$

where $s_{ij} = \sum_{k=1}^{n} [x_{ki} - \bar{x}_i][x_{kj} - \bar{x}_j]/n$, $i = 1,2$, $j = 1,2$. Find the area of the parallelogram formed by $\mathbf{e}_1$ and $\mathbf{e}_2$ and relate it to $|S|$, the determinant of S.
(8). If $n = 10$ and the lengths of $\mathbf{e}_1$ and $\mathbf{e}_2$ are 5 and 10, respectively. Find S.