國立成功大學 80學年度 統計所考試(機率論 試題)

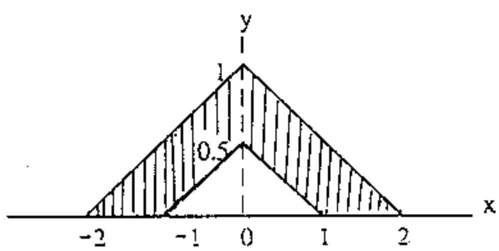
Probability Theory

I. (25 points) Multiple choice. Choose the **mose appropriate answer** for the following questions.

(a). Assume that X and Y are independent random variables, each with geometric didtribution $P(X = n) = pq^{n-1}$, $P(Y = n) = \alpha\beta^{n-1}$, $n = 1,2,\ldots$ with $q = 1 - p$, $\beta = 1 - \alpha$. Then $P(X=Y)$ is

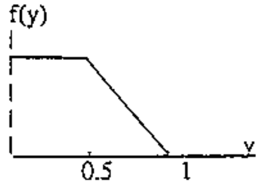
(1). $\sum_{n=1}^{\infty}(p\beta)^{n-1}q\alpha$ (2). $\sum_{n=1}^{\infty}(q\alpha)^{n-1}p\beta$ (3). $\frac{p\alpha}{(1-q\beta)}$ (4). None of the above.

(b). Random variables X and Y have joint density that is constant on the shaded area below:

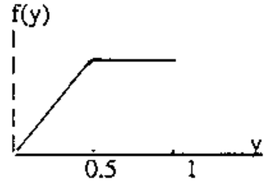


Then the marginal density of Y looks like

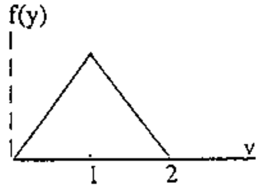
(1).



(2).



(3).



(4). None of the above

(c). Assume that X, Y, Z are independent random variables, with probability distribution functions

$F(t) = P(X \le t)$, $G(t) = P(Y \le t)$, $H(t) = P(Z \le t)$

Let $U = \min\{X,Y,Z\}$. Then the probability distribution function of U is

(1). $\mathrm{Min}\{F(t), G(t), H(t)\}$, (2). $F(t) \cdot G(t) \cdot H(t)$

(3). $1 - (1 - F(t))(1 - G(t))(1 - H(t))$, (4). None of the above.

(d). In comparing $P(A \cap B)$ and $P(B)$

(1). We have $P(A \cap B) = P(B)$ only if $A \cap B = \phi$,
(2). We always have $P(A \cap B) < P(B)$
(3). It may happen that $P(A \cap B) > P(B)$,
(4). We have $P(A \cap B) = P(B)$ if $A \supset B$.

(e). The distribution of the sum of two independent variables drawn from the same uniform distribution is:

(1). Uniform, (2). Triangular, (3). Normal, (4). hypergeometric.

II. (25 points) Let (X,Y) be a bivariate random variable with the following joint probability distribution:

X \ Y	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2
2	0	0	1/8	0	0
$\sqrt{2}$	0	1/8	0	1/8	0
0	1/8	0	0	0	1/8
$-\sqrt{2}$	0	1/8	0	1/8	0
-2	0	0	1/8	0	0

(a). Find E(X) and E(Y).
(b). Find Var(X) and Var(Y).
(c). Show that the correlation coefficient of X and Y is 0.
(d). Note that $P(X^2 + Y^2 = 4) = 1$, but in (c) we have $\rho_{xy} = 0$. Why?
(e). If we only know the marginal distributions of X and Y, can we uniquely determine the distribution of (X,Y)? Use the above joint probability distribution to support your argument.

III. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Let $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$, then the mean vector and variance-covariance matrix of **X** are $E(\mathbf{X}) = \boldsymbol{\mu}$, $Cov(\mathbf{X}) = \sigma^2\mathbf{I}$, respectively, where $\boldsymbol{\mu} = [\mu, \mu, \ldots, \mu]^T$ is an $n \times 1$ column vector, **I** is an $n \times n$ identity matrix. (25 points)

(a). What is the distribution of $X_i - \bar{X}$? (9 points)

(b). It is well known that if **a** and **b** are two $n \times 1$ column vectors, then

$Cov(\mathbf{a}^T\mathbf{X}, \mathbf{b}^T\mathbf{X}) = \mathbf{a}^T Cov(\mathbf{X})\mathbf{b}$. Use this fact and normal theory to prove that $X_i - \bar{X}$

and $\bar{X}$ are statistically independent. (9 points) (c). Cite one theorem (for example, from the book of Hogg & Craig) to argue that

$\sum_{i=1}^{n}[X_i-\bar{X}]^2/n$ and $\bar{X}$ are statistically independent. (7 points)

IV. Let X be a continuous random variable with probability density function $f_X(x)$. Define

$$U = F(X) = \int_{-\infty}^{X} f_X(y)dy.$$

(a). Show that U is uniformly distributed over the interval (0, 1). (10 points)
(b). Suppose we take a random sample $X_1,X_2,\ldots,X_n$ from $f_X(x)$ and let

$\lambda(X_1,X_2,\ldots,X_n) = \prod_{i=1}^{n} F(X_i)$. Find the distribution of $-2\ln\lambda(X_1,X_2,\ldots,X_n)$, where $\ln X$ is the natural logarithm of X. (5 points)
(c). Find $E[-2\ln\lambda(X_1,X_2,\ldots,X_n)]$ and $Var(-2\ln\lambda(X_1,X_2,\ldots,X_n))$. (5 points)
(d). What is the limiting distribution of $-\ln\lambda(X_1,X_2,\ldots,X_n)/\sqrt{n} - \sqrt{n}$? (5 points)