CALCULUS: 50 points

(1) Evaluate the following integrals

(a) $\int_0^1 \int_y^1 (\sin x^2 + \exp (- x^2)) \, dx \, dy$ (5 points)

(b) $\iint_{R^2} e^{-(x^2 + y^2)} \left(1 + \frac{xy}{(1 + x^2)(1 + y^2)} \right) dxdy$ (5 points)

(2) Consider the function defined by

$$G(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2 + y^2} & \text{if } x^2 + y^2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Are $G_{xy}(0,0)$ and $G_{yx}(0,0)$ equal ? (10 points)

(3) Let $F(x,y,z) = x e^{yz} + 2z$ and $P = (2,0,-4)$

(a) Find the path of steepest ascent at point P. (5 points)

(b) Let S be the surface given by $F(x,y,z) + 6 = 0$. Find the equation of the tangent plane to S at P. (5 points)

(c) Find the angle θ, $0 \le \theta \le \pi/2$, between the tangent plane and the xy-plane. (5 points)

(4) Evaluate the limit, if it exists

(a) $\lim_{x \to 0+} (e^x + x)^{1/x}$ (3 points)

(b) $\lim_{(x,y) \to (0,0)} \frac{4x^2y}{x^4+y^2}$ (3 points)

(c) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i\,!}$, where x ! is factorial of x. (3 points)

(5) Prove or disprove that the existence of the partial derivatives F_x and F_y at (x_0,y_0) is sufficient to guarantee differentiability of F at the point. (6 point)

Linear Algebra: 50 points

(1). Let A be an $n \times n$ (real) symmetric matrix. Prove or disprove

(a) The eigenvectors of A are orthogonal if all the eigenvalues are distinct. (5 points)

(b) If two eigenvectors are orthogonal, then the corresponding eigenvalues are distinct. (5 points)

(2). Let B be an $n \times n$ (real) symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated normalized eigenvectors $e_1, \ldots, e_n$.

(a) Show that

$$\max_{x \neq 0} \frac{x^t B x}{x^t x} = \max_j \{ \lambda_j \} \qquad \text{(10 points)}$$

where x^t is the transpose of vector x.

Also, show that the maximum is attained when $x = e_i$ for some i.

(b) Assume $B = (\rho_{ij})$ with $\rho_{ij} = 1$ if $i = j$ and $= \rho$ if $i \neq j$, and $|\rho| < 1$. Use (a) to compute

$$\max_{x \neq 0} \frac{x^t B x}{x^t x} \qquad \text{(5 points)}$$

(c) Determine the required range for ρ in (b) so that B is a positive definite matrix. (5 points)

(3) Let $b = (2, 1, 5)$, $a_1 = (1, 2, 1)$, and $a_2 = (2, 1, -1)$. Also, let W be the subspace generated by a_1 and a_2.

(a) Find the projection of b on W. (5 points)

(b) What is the projection matrix for W? (5 points)

(4) Let $D = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ and define $\exp(D) = \lim_{m \to \infty} \left(I + D + \cdots + \frac{D^m}{m!} \right)$

Compute exp(D). (10 points)