Probability Theory
(contain mathematical statistics)

(10%) 1. Let X and Y be i.i.d. random variables from the geometric distribution. Define U=X-Y and V=X+Y. Find
(a) the distribution of U.
(b) the join moment generating function of X and V.

(20%) 2. Let X and Y be i.i.d. random variables from exponential distribution with parameter $\lambda$. Define U=X(1+Y) and V=Y. Find
(a) $P(1<X+Y<2)$.
(b) $P(X<Y|X<2Y)$.
(c) $E(U|V)$
(d) the covariance of U and V.

(10%) 3. Let X be a continuous random variable with median m. Show that $E(|X-b|)$ is minimized when b=m.

(10%) 4. Let $X_1,\ldots,X_n$ are i.i.d. random variables with c.d.f.

$$F(x)=(1-e^{-\lambda x})I_{(0,\infty)}(x), \text{ where } \lambda>0 .$$

Define $Y_n=\max[X_1,\ldots,X_n]$. Find
(a) $E[F(Y_n)]$.
(b) the limiting distribution of $\lambda(Y_n-\frac{1}{\lambda}\log(n))$.

(30%) 5. Let $X_1,\ldots,X_n$ be i.i.d. random variables from the Uniform distribution$(0,\theta)$. Let $Y_1<\ldots<Y_n$ denote the corresponding order statistics.
(a) Estimate $\theta$ by the method of moments.
(b) Find the maximum-likelihood estimator of $\theta$.
(c) Among all estimators of the form $cY_n$ , where c is a constant which may depend on n, find that estimator which has uniform -ly smallest mean-squared error.
(d) Find the UMVUE of $\theta$.
(e) Let $T=Y_1+Y_n$. Find the mean and mean-squared error of T.
(f) Which one would you use in the above estimators and why?

(20%) 6. Let X be a single observation from the density

$$f(x;\theta)=\theta x^{\theta-1} I_{(0,1)}(x), \text{ where } \theta>0 .$$

(a) To test $H_0:\theta\leq 1$ versus $H_1:\theta>1$, find the power fuction and size of the test given by the following procedure : Reject $H_0$ if $X\geq 1/2$ .
(b) Find the most powerful size-$\alpha$ test of $H_0:\theta=2$ v.s. $H_1:\theta=1$.
(c) Find the likelihood-ratio test of size $\alpha$ of $H_0:\theta=1$ v.s. $H_1:\theta\neq 1$ .
(d) Given a set of observation, all of which fall between 0 and 1, indicate how you would test the hypothesis that the obser -vations came from the density $f(x;\theta)$.

358