(6%) 1. Let Y be a random variable having a Poisson distribution with parameter λ. Assume that the conditional distribution of X given $Y = y$ is binomially distributed with parameters y and p. Find $E(X)$ and $Var(X)$.

2. If X and Y are independent random variables, each having the same exponential distribution with c.d.f.

$$F(x) = 1 - e^{-\lambda x}$$

for $x > 0, \lambda > 0$.

(5%)(a) Find the distribution of $F(X)$.

(6%)(b) Find $P(X < Y)$.

(6%)(c) Find the joint p.d.f. of $X/(X+Y)$ and $X+Y$.

(6%)(d) Define $U = \min(X, Y)$ and $V = \max(X, Y)$. Find the p.d.f. of U.

(6%)(e) Find $E[V]$.

(8%)(f) Let W be independent of X and Y, and assume that W is Bernoulli distributed with parameter ρ.
Define $Z = WX + (1 - W)Y$. Find the correlation coefficient of X and Z.

(5%)(g) Find the distribution of Z.

(6%) 3. Assume that $X_1, \ldots, X_{k+1}$ are independent Poisson distributed random variables with respective parameters $\lambda_1, \ldots, \lambda_{k+1}$, Find the conditional distribution of $X_1, \ldots, X_k$ given that $X_1 + \cdots + X_{k+1} = n$.

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables from the c.d.f

$$F(x) = 1 - x^{-\theta}$$

for $x > 1, \theta > 0$. Find

(5%)(a) the maximum likelihood estimator(MLE) of $1/\theta$.

(5%)(b) the Cramér-Rao lower bound for unbiased estimators of $1/\theta$.

(6%)(c) the UMVUE of $1/\theta$.

(6%)(d) the UMVUE of θ.

(8%)(e) the limiting distribution of $\sqrt{n}(\log \hat{\theta}_n - \log \theta)$, where $\hat{\theta}_n$ is the MLE of θ.

5. Let $X_1, \ldots, X_n$ be i.i.d. random variables from a exponential distribution with p.d.f.

$$f(x) = \theta e^{-\theta x}$$

for $x > 0, \theta > 0$.

(6%)(a) Derive the most powerful size-α test of $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_1$, where $\theta_1 > \theta_0$.

(6%)(b) Derive the generalized likelihood ratio(GLR) test of size α of $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$.

(4%)(c) Compute the power of the test in (a).