注意:未寫明演算過程者不予計分

1. Let $F(x,y,z) = 3xy - y^2 + z^2$. Let $P = (1,0,3)$ and $Q = (7,2,6)$. (10%)

   (a) Find a unit vector $\mathbf{u}_1$ such that $F(x,y,z)$ increases most rapidly as on leaves from P in the direction $\mathbf{u}_1$. Why?

   (b) Let $\mathbf{u}_2$ be a unit vector in the direction from P to Q. Find the directional derivative $D_{\mathbf{u}_2}F(P)$. Is F increasing or decreasing the instant one leaves from P going toward Q?

2. Find the surface area and the volume of the ellipsoid(橢球) obtained by revolving an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis. (10%)

3. Find the following values, if exist. (20%)

   (a) $\int_0^\infty y^2 e^{-\sqrt{y}}dy$ (b) $\lim_{x\to 0}(1-4x^2)^{3/x}$

   (c) $\int_0^1 x^5(1-x^2)^{\frac{10}{3}}dx$ (d) $\int \sqrt{9-4x^2}dx$

4. For the given power series $\sum_{k=1}^{\infty} \frac{\ln k}{e^k}(x-e)^k$, find the interval of convergence. (5%)

5. Determine whether the following statements are true or false. (5%)

   (a) A conditionally convergent series may diverges.

   (b) If $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, then $\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.

   (c) Let $\rho_k = |\frac{a_{k+1}}{a_k}|$. If $\lim_{k\to\infty} \rho_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges.

   (d) If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then $\sum_{k=1}^{\infty} {a_k}^2$ converges.

   (e) If Taylor's series of $f(x)$ converges, then it converges to $f(x)$.

(背面仍有題目,請繼續作答)

6. Let (10%)

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}.$$

Determine an orthogonal matrix T and a diagonal matrix B such that $B = T^{-1}AT$.

7. Let A be a square matrix. If a matrix X such that $AXA = A$, then this X is called a generalized inverse of A. If A satisfies $(A - I)^3 = A - I$. Find a generalized inverse of A. (5%)

8. Let S be the set of all real 2×2 symmetric matrices. (15%)

(a) Is S a vector space? If yes, find a basis for S.

(b) Show that tr is a linear transformation from S to the real numbers, where $tr(A) = trace\ of\ A$. And find the null space of tr.

(c) For $A,\ B \in S$, does $< A, B >= tr(AB)$ define an inner product on S? Justify your answer.

9. Let f be the bilinear form on R^2 defined by (10%)

$$f((x_1, x_2), (y_1, y_2)) = 2x_1y_1 - 3x_1y_2 + x_2y_2.$$

(a) Find the matrix A of f in the basis $\{u_1 = (1,0), u_2 = (1,1)\}$.

(b) Find the matrix B of f in the basis $\{v_1 = (2,1), v_2 = (1,-1)\}$.

(c) Find the transition matrix P from the basis $\{u_i\}$ to the basis $\{v_i\}$, and verify the equation which states the relation between A, B and P.

10. Let A be a symmetric positive definite matrix and let x, y denote $n \times 1$ vectors. (10%)

(a) Show that

$$x^tA^{-1}x = \sup_y \frac{(x^ty)^2}{y^tAy}.$$

(b) Use part (a) to show that if A_1 and A_2 are symmetric positive definite matrices and $0 < \alpha < 1$ then

$$(\alpha A_1 + (1-\alpha)A_2)^{-1} \leq \alpha A_1^{-1} + (1-\alpha)A_2^{-1},$$

where $A \leq B$ means $A - B$ is positive semi-definite.