注意：未寫明演算過程者不給分!

1. Suppose that the function $f(x)$ is such that $f'(x)$ and $f''(x)$ are continuous in a neighborhood of the regin and satisfies $f(0) = 0$. Show that

$$\lim_{x \to 0} \frac{d}{dx}[\frac{f(x)}{x}] = \frac{1}{2}f''(0) \qquad (5\text{分})$$

2. Suppose that $g : R \to R$ and that $|g'(x)| < M$ for all $x \in R$, where M is a positive constant. Define $f(x) = x + cg(x)$, where c is a positive constant. Show that it is possible to choose c small enough so that f is a one-to-one function. (6分)

3. Suppose that $f(x)$ is continuous on $[0, \infty)$, $f''(x)$ exists on $(0, \infty)$. Show that $g(x)$ is monotone increasing on on $(0, \infty)$ where $g(x) = f(x)/x$. (7分)

4. Suppose that we have the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_1 = 1$ and

$$a_{n+1} = \frac{a_n(3b + a_n^2)}{3a_n^2 + b}, \qquad b > 0, n = 1, 2, ...$$

Show that the sequence converges and find its limit. (10分)

5. Consider the improper integral $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$. Show that

$$(a) B(m, n) = 2\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta d\theta \qquad (4\text{分})$$

$$(b) B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad (4\text{分})$$

$$(c) B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}dx \qquad (4\text{分})$$

6. Find

(a) $\int\int_D (x^2 + y^3)dxdy$, where D is the region in the first quadrant bounded by $y = x^2$ and $x = y^4$ (5分)

(b) $\int\int_D xy^2 dxdy$, where D is bounded by the four parabolas, $y^2 = x$, $y^2 = 3x$, $x^2 = y$ and $x^2 = 4y$. (5分)

(背面仍有題目,請繼續作答)

7. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ idempotent matrices. Show that $\mathbf{A} - \mathbf{B}$ is idempotent if and only if $\mathbf{AB} = \mathbf{BA} = \mathbf{B}$. (10 分)

8. The matrices $\mathbf{A}$ and $\mathbf{B}$ are systematic and positive semidefinite of order $n \times n$ such that $\mathbf{AB} = \mathbf{BA}$. Show that $\mathbf{AB}$ is positive semidefinite. (10 分)

9. Let $\mathbf{L} : R^n \to R^m$ be a linear transformation defined by $\mathbf{L}(\mathbf{X}) = \mathbf{AX}$, $\mathbf{X}$ in R^n, where $\mathbf{A}$ is an $m \times n$ matrix. Prove that

(a) $\mathbf{L}$ is one-to-one if rank $\mathbf{A} = n$. (5 分)

(b) $\mathbf{L}$ is onto if rank $\mathbf{A} = m$. (5 分)

10. If $\mathbf{A}$ be an $n \times n$ systematic matrix, and let λ be an eigenvalue of $\mathbf{A}$ of multiplicity k, then $\mathbf{A} - \lambda \mathbf{I}_n$ has rank $n - k$. (10 分)

11. (a) Show that $tr(\mathbf{A}^T\mathbf{A}) = 0$ if and only if $\mathbf{A} = \mathbf{0}$. (5 分)

(b) If $\mathbf{A}$ is a systematic $n \times n$ matrix, and $\mathbf{B}$ is an $n \times n$ skew-systematic matrix, then show that $tr(\mathbf{AB}) = 0$. (5 分)