注意:未寫明演算過程者不予計分

1. Find the following values, if exist.

(5%) (a) $\lim_{x\to a} \frac{\sin^2(x-a)}{(x-a)^2}$.

(5%) (b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$.

(5%) (c) $\lim_{n\to\infty} \int_0^{2\pi} \frac{\sin nx}{x^2+n^2}dx$.

(5%) (d) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$.

(5%) (e) $\int_0^{\infty} x^{-1/2}e^{-x}dx$.

(5%) (f) $\int_0^{\infty} e^{-st}\frac{\sin t}{t}dt$.

2. Test for convergence or divergence and explain it.

(3%) (a) $\sum_{k=2}^{\infty}[\sin^{-1}(\frac{1}{k})]^k$.

(3%) (b) $\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k}$.

(4%) (c) For what values of x does $\sum_{k=1}^{\infty}\frac{1}{(x+k)(x+k-1)}$ converge?

3. (5%) (a) Find the volume of one of the wedges cut from the cylinder $4x^2+y^2=a^2$ by the planes $z=0$ and $z=my$.

(5%) (b) $\int_1^2 dx \int_x^{x^3} e^y\sqrt{\frac{x}{y}}dy + \int_2^8 dy \int_2^y e^y\sqrt{\frac{x}{y}}dx = ?$

(10%) 4. If A is a $n\times n$ symmetric matrix. Prove that A is idempotent and of rank r if and only if it has r eigenvalues equal to 1 and $n-r$ eigenvalues equal to 0.

5. Let A be a $n\times n$ positive definite matrix with eigenvalues $\lambda_1>\lambda_2>\dots>\lambda_n$ and associated normalized eigenvectors $\vec{e}_1,\vec{e}_2,\dots,\vec{e}_n$.

(6%) (a) Prove that

$$\max_{\vec{x}\neq 0}\frac{\vec{x}'A\vec{x}}{\vec{x}'\vec{x}}=\lambda_1$$

attained when $\vec{x}=\vec{e}_1$.

(4%) (b) Let $A^{1/2}=\sum_{i=1}^{n}\sqrt{\lambda_i}\vec{e}_i\vec{e}_i'$ be the square root matrix of A.
Show that $(A^{1/2})^{-1}=\sum_{i=1}^{n}\frac{1}{\sqrt{\lambda_i}}\vec{e}_i\vec{e}_i'$.

6. Let

$$A=\begin{bmatrix}3&1&0\\1&3&0\\0&0&1\end{bmatrix}$$

(5%) (a) Show that A is positive definite matrix.

(5%) (b) Find the square root matrix $A^{1/2}$.

(5%) (c) Let A^{-1} be the inverse of A.
Find the eigenvalues and eigenvectors of A^{-1}.

(5%) (d) Find the spectral decomposition of A^{-1}.

(4%) (e) Transform the equation $3x_1^2+3x_2^2+x_3^2+2x_1x_2-4=0$ to be an ellipse equation $\frac{y_1^2}{a^2}+\frac{y_2^2}{b^2}+\frac{y_3^2}{c^2}=1$.

(6%) (f) Let $|A|$ be the determinant of A. Find $|A^5|$ and the trace of A^5.

7/30