1. State which of the following statements are true and which are false. (20 pts)

____ (a) If $f''(x)$ exists then f' is continuous at x. (f' is the derivative of f)

____ (b) If f is continuous and has a maximum at $x = x_0$, then $f'(x_0) = 0$.

____ (c) If $f'(x) = g'(x)$, $\forall$ x, then $f(x) = g(x)$.

____ (d) If $\int_a^b f(x)dx = 0$ then $f(x) = 0$.

____ (e) If $ln\, a = ln\, b$ then $a = b$. (ln means natural logarithm)

____ (f) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

____ (g) If $\{a_n\}$ is a monotone decreasing sequence of positive numbers, then the sequence converges.

____ (h) If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and $\frac{a_{n+1}}{a_n} < 1$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges

____ (i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ may converge or diverge.

____ (j) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\left(\frac{f}{g}\right)'(x) = 0$.

Please show all work ! (下列各題請列詳細演算過程)

2. Evaluate the following derivatives or limits. (8 pts)

(a) $D_x\, ln|\cos x|$

(b) $D_x\, (\sin x)^x$, for $0 < x < \pi$.

(c) $\lim_{x\to 0^+} (\sin x)^x$

(d) $\lim_{x\to 0^+} (\sinh x)^x$

3. Evaluate the following Integrals. (8 pts)

(a) $\int e^{2x} \cos 3x\, dx$

(b) $\int_0^{1/2} \sqrt{1-u^2}\, du$

(c) $\int \frac{2x-3}{x^5 + 2x^3 + 3}\, dx$

(d) $\int_0^{\infty} e^{-x^2/2}\, dx$

4. (a) Prove that $\lim_{n\to\infty}(1+h)^{1/h}=e$ (5 pts)

(b) Evaluate $\lim_{n\to\infty}\left(1-\frac{3}{n}\right)^{2n}$ (2 pts)

(c) Evaluate $\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{n/2}$ (2 pts)

5. Prove Wallis's formula for n>1 : (5 pts)

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases} \dfrac{2\cdot4\cdot6\cdot\cdots\cdot(n-1)}{1\cdot3\cdot5\cdot\cdots\cdot n} & \text{if } n \text{ is odd} \\[2ex] \dfrac{1\cdot3\cdot5\cdot\cdots\cdot(n-1)}{2\cdot4\cdot6\cdot\cdots\cdot n}\,\dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

6. True or False, please give a reason or counterexample (15 pts)

(a) If an n x n matrix A has n distinct eigenvalues if and only if A is diagonalizable.

(b) If λ is an eigenvalue of matrix A and β is an eigenvalues of matrix B then $\lambda\beta$ is also an eigenvalue of AB.

(c) If A is invertible and A^{-1} is symmetric then A is symmetric matrix too.

(d) If A and B are n x n positive definite matrices then A+B is positive definite matrix.

7. Let

$$A=\begin{bmatrix} n & n_1 & n_2 & n_3 & n_4 \\ n_1 & n_1 & 0 & 0 & 0 \\ n_2 & 0 & n_2 & 0 & 0 \\ n_3 & 0 & 0 & n_3 & 0 \\ n_4 & 0 & 0 & 0 & n_4 \end{bmatrix}$$

where n_1, n_2, n_3, and n_4 are positive integers and

$n = n_1+n_2+n_3+n_4$.

(a) Find the determinant of A. (4 pts)

(b) Find the rank of A. (4 pts)

(c) Find the dimension of the nullspace of A. (3 pts)

(d) Is A diagonalizable ? (2 pts)

(e) Is A a projection matrix ? (2 pts)

8. Let V be the space of all real polynomials with degrees not exceeding 3, and

$$f \cdot g = \int_0^1 f(x)g(x)\,dx \qquad (f, g \in V).$$

Show that

$$\left(\int_0^1 f(x)g(x)\,dx\right)^2 \le \int_0^1 f^2(x)dx \int_0^1 g^2(x)dx. \qquad \text{(5 pts)}$$

9. Find the maximum and minimum values for the quadratic form (10 pts)
$f(x, y, z) = 4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz$,
subject to the constraint $x^2 + y^2 + z^2 = 1$.

10. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of matrix A, show that the determinant of A is $\prod_{i=1}^{n} \lambda_i$

(5 pts).