I. (10 points) It is well-known by central limit theorem that for *i.i.d.* random sample $X_1, X_2, \ldots, X_n$, with $E(X_1^2) < \infty$, the random variable $\sqrt{n}(\overline{X} - E(X_1))$ converges in distribution to $N(0, Var(X_1))$. Suppose we have $X_1, X_2, \ldots, X_n$, *i.i.d.* from Cauchy(0, 1), i.e.,

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in R,$$

It is known that the characteristic function of a Cauchy random variable is $\phi(t) = E[e^{itX}] = e^{-|t|}$. Prove that the central limit theorem can not by applied to Cauchy by showing that $\overline{X} = \sum_{i=1}^{n} X_i / n$ has the same distribution as X_1.

II. (30 points) Consider the following two sampling experiments:

a. Conduct n i.i.d. Bernoulli trials, say, $X_1, X_2, \ldots, X_n$ with probability of success eqaul to $E(X_i)=p$, $0<p<1$.

b. The experiment is continued until a specified number of successes have been obtained. (Each Bernoulli trial is i.i.d. with probability of success equal to p, too.) Let Y_i denote the number of trials after the (i-1)th success up to but not including the ith success.

(1). Find the joint distributions of $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$, respectively, and then find the sufficient statistics for the experiments (a) and (b).

(2). Consider the estimation of p and $1/p$. Find the UMVU estimators of p for experiment (a) and of $1/p$ for the experiment (b), respectively.

(3). Find the Cramer-Rao lower bound (CRLB) for p in (a) and $1/p$ in (b). Do the variances of the UMVU estimators coincide with the CRLBs ?

III. (60 points) Let (Y, X) be a bivariate random vector with P(Y=1)=1-P(Y=0)=p, and X|Y=i follows N(μ_i, σ^2), i=0, 1, $\mu_1 > \mu_0$, σ^2 known.

(1). Find P(Y=i|X=x), i=0,1.

(2). Suppose μ_0, μ_1 and p are known, and an observation X=x is given. We want to classify x to one of the two groups Y=1 or 0 based on which of the posterior probability P(Y=i|X=x), i=0, 1, is larger. Show that the decision rule can be written as a linear function of x.

(3). For this (3) only. Let μ_1=2, μ_0=1, σ^2=1, p =1/2. Find the error probabilities

P(classified as Y= i |X is from Y=1- i), i =1,0.

Suppose now we have the classical setting, i.e.,

$$f_i(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu_i}{\sigma^2})}, \quad i = 0,1, \quad x \in R,$$

where μ_i, σ, as before, are known, and we want to determine which population x is coming form.

(4). Find a most powerful level 0.05 test for $H_0: f_0$ vs $H_1: f_1$.

(5). Is there any connection between the rules you found in (2) and (4) ? Discuss.

(6). We say (Y, X) has a logistic regression model if the conditional expectation E(Y|X=x) has the following form

$$E(Y|X=x) = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x)}, \qquad (*)$$

where Y is binary with P(Y=1)=p, 0<p<1. Suppose X given Y=i has pdf

$$f(x|Y=i) = e^{x\theta_i + d(\theta_i)} h(x), \ i=0,1.$$

Show that E(Y|X=x) has the form of (∗) and hence deduce that the rule in (2) falls into the framework of logistic regression model.