注意：未寫明演算過程者不予計分。

(10%) (一) State which of the following are true and which are false. If false, please correct it.(若答案爲錯，則請更正)

(1) In polar coordinate system, the slope of the equation $r = f(\theta)$ at a point $(r, \theta)$ on the graph is $f'(\theta)$.

(2) If $\{a_n\}$ is a monotone increasing sequence of positive numbers, then the sequence $\{a_n\}$ converges.

(3) If $f$ is continuous at $(x_0, y_0)$, and $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist, then $f$ is differentiable at $(x_0, y_0)$.

(4) If $\nabla f(x_0, y_0) = 0$, then $f$ has either a local maximum or a local minimum at $(x_0, y_0)$.

(5) If $f$ is defined on the rectangular region $R = \{(x, y): a \le x \le b,\ \ c \le y \le d\}$, then $\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx$.

(10%) (二)

a) Find the area between the circles $r = 1$ and $r = 2\cos\theta$.

b) Evaluate the integral $\iint_R x\sqrt{x^2 + y^2}\, dA$ where $R$ is the disk with its center at the origin and radius 1.

(10%) (三) Evaluate $\iint_R (9x - 3y)\, dA$ where $R$ is the region bounded by $3x - y = 1$, $3x - y = 3$, $x + y = 1$, $x + y = 2$.

(10%) (四)

a) Suppose that $\Sigma a_n$ and $\Sigma b_n$ are series of positive terms. Prove that if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\Sigma b_n$ converges, then $\Sigma a_n$ also converges.

b) Determine convergence or divergence of the following series.

(i) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ (ii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$

(10%) (五)

a) Let $f(x) = xe^x$. Make use of power series and $f'(1)$ to find the sum of the series $\sum_{n=0}^{\infty} \frac{n+1}{n!}$.

b) Prove that $\lim_{x\to 0} (1 + x)^{1/x} = e$.

(背面仍有題目,請繼續作答)

134001

(10%) (六) Let $R$ be the space of real numbers. Define $T: C[a,b] \to R$ by

$T(f) = \int_a^b f(x)\,dx$, where $C[a,b]$ is the set of functions continuous on [a, b].

Using element properties of integrals, prove that $T$ is a linear transformation.

(10%) (七) Produce a matrix $P$ such that $P^{-1}AP$ is diagonal, where $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$.

(10%) (八)

(a) Let $\lambda_1$ and $\lambda_2$ be distinct eigenvalues of the real, symmetric matrix A. Suppose that $v_1$ and $v_2$ are associated eigenvectors. Prove that $v_1$ and $v_2$ are orthogonal.

(b) Let $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$. Use the eigenvectors of A to construct an orthogonal matrix $P$ i.e. $P^{-1} = P^t$.

(10%) (九) A real, symmetric matrix is positive definite if every eigenvalue is positive.

(a) Let A be a real symmetric matrix. Prove that A is positive definite if and only if there is a nonsingular matrix $Q$ such that $A = Q^tQ$.

(b) Let $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$. Find a nonsingular matrix $Q$ such that $A = Q^tQ$.

(10%) (十)

(a) Find the canonical form of $x_1^2 + 2x_2^2 + 2\sqrt{2}x_1x_3$.

(b) Let $X = (x_1, x_2, x_3)^t$, $Y = (y_1, y_2, y_3)^t$. Find a matrix $P$ such that $X = PY$.

transform $x_1^2 + 2x_2^2 + 2\sqrt{2}x_1x_3$ into its canonical form.