壹．選擇題：下列各題均只有一個最佳答案，每題四分，共 24 分。

1. Which of the following statements is(are) true?
   (i) If E is independent of F and E is independent of G, then E is independent of $F \cup G$
   (ii) If E is independent of F and E is independent of G, and $FG = \phi$, then E is independent of $F \cup G$
   (iii) If E is independent of F, and F is independent of G, and E is independent of FG, then G is independent of EF
   (a) i (b) ii (c) iii (d) i and ii (e) i and iii (f) ii and iii (g) i, ii and iii

2. If Z is a standard normal random variable, then for every $a > 0$, $\lim_{x\to\infty} \frac{P\{Z \ge x + a/x\}}{P\{Z \ge x\}} = ?$
   (a) 1 (b) e^{-a} (c) e^{a} (d) $e^{-a/2}$ (e) $e^{a/2}$ (f) $e^{-a^2/2}$ (g) $e^{a^2/2}$

3. A man and a woman decide to meet at a certain location. If each person independently arrives at a time uniformly distributed between 12 noon and 1 p.m., then the probability that the first to arrive has to wait longer than 10 minutes is
   (a) 1/4 (b) 1/3 (c) 5/12 (d) 1/2 (e) 5/9 (f) 25/36 (g) 5/6

4. If (I) X and Y are independent Poisson random variables and
   (II) X and Y are independent binomial random variables
   calculate the conditional distribution of X, given X+Y=n, respectively, we conclude that
   (a) both (I) and (II) are binomial distribution
   (b) both (I) and (II) are hypergeometric distribution
   (c) case(I) is binomial distribution and case (II) is hypergeometric distribution
   (d) case(I) is hypergeometric distribution and case (II) is binomial distribution
   (e) case(I) is Poisson distribution and case (II) is binomial distribution
   (f) case(I) is Poisson distribution and case (II) is hypergeometric distribution
   (g) none of them

5. If $E[Y \mid X = x] = E[Y]$ for all x, then which of the following statements is(are) true?
   (i) X and Y are uncorrelated (ii) X and Y are independent (iii) the converse is not true
   The correct answer is (a) i (b) ii (c) iii (d) i and ii (e) i and iii (f) ii and iii (g) i, ii and iii

6. Consider a biased coin where the probability of a head, p, is known to be 0.2, 0.3 or 0.8. The coin is tossed repeatedly, and we let X be the number of tosses required to obtain the first head. To test $H_0 : p = 0.8$, suppose we reject H_0 if $X \ge 3$, and do not reject otherwise. Compute the probability of Type I error α, the probability of Type II error β_1 for $p = 0.2$ and the probability of Type II error β_2 for $p = 0.3$, we have $(\alpha, \beta_1, \beta_2) =$
   (a) (0.04, 0.36, 0.60) (b) (0.04,0.36,0.51) (c) (0.04,0.2,0.3) (d) (0.04,0.36,0.3)
   (e) (0.05,0.36,0.59) (f) (0.05,0.36,0.51) (g) (0.05,0.2,0.3)

(背面仍有題目,請繼續作答)

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貳. 計算與證明題：共 76 分

1. Along a road 1 mile long are 3 people "distributed at random" (the positions of the 3 people are independent and uniformly distributed over the road). Find the probability that no 2 people are less than a distance of d miles apart, when $d \le 1/2$. (10%)

2. If X and Y are independent gamma random variables with parameters (α, λ) and (β, λ), respectively, the p.d.f. of X is $f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, 0 < x < \infty$, compute the joint density of U=X+Y and V=X/(X+Y). Are U and V independent? (10%)

3. A group of N men throw their hats into the center of a room. The hats are mixed up, and each man randomly selects one. Find
   (a) the expected number of men that select their own hats,
   (b) the variance of the number of men that select their own hats. (10%)

4. If X is a random variable with mean 0 and finite variance σ^2, Prove that
   (a) for any random variables $Y, (E[XY])^2 \le E[X^2]E[Y^2]$,
   (b) for any $a > 0$, $a \le \int_{-\infty}^{\infty}(x-a)I_a(x)dF(x)$, where $F(x)$ is the distribution function of X and $I_a(x) = 1$, if $x \le a$; $I_a(x) = 0$, if $x > a$,
   (c) for any $a > 0$, $p\{X > a\} \le \dfrac{\sigma^2}{\sigma^2 + a^2}$ (14%)

5. Let Y_2 be the second order statistic of a random sample of size n, $n=2$, from a continuous-type uniform distribution on the interval$(0, \theta)$. Let $0 < c_1 < c_2 \le 1$ be selected so that $\Pr(c_1\theta < Y_2 < c_2\theta) = 0.95$. Base on this equation and Y_2, find c_1, c_2 and the 95% confidence interval of θ such that the confidence interval has the shortest length. (10%)

6. Consider a random sample of size n from an exponential distribution with p.d.f.
   $f(x;\theta) = \dfrac{1}{\theta}e^{-x/\theta}, 0 < x < \infty$, and given that $\sum_{i=1}^{n}\dfrac{2X_i}{\theta}$ distributed as chi-squared distribution $\chi^2(2n)$ with degrees of freedom $2n$. Find
   (a) the MLE (maximum likelihood estimator) of θ,
   (b) the UMVUE (uniformly minimum variance unbiased estimator) of θ,
   (c) the CRLB (Cramer-Rao lower bound) of θ,
   (d) the 95% confidence interval of θ,
   (e) the GLR (Gnerealized likelihood ratio) test of $H_0: \theta = \theta_0$ versus $H_1: \theta > \theta_0$ (22%)