1. Let X and Y denote two binary random variables. Let $P(Y=j \mid X=i)$ be conditional probability, with i, j = 0,1. Define $\alpha = \dfrac{P(Y=1 \mid X=1)}{P(Y=0 \mid X=1)}$, and $\beta = \dfrac{P(Y=1 \mid X=0)}{P(Y=0 \mid X=0)}$. Let $\theta = \dfrac{\alpha}{\beta}$.

   (1) Show that if X and Y are independent then $\theta = 1$. (2 pts)

   (2) Show that θ can be expressed as $\dfrac{P(X=1 \mid Y=1)P(X=0 \mid Y=0)}{P(X=0 \mid Y=1)P(X=1 \mid Y=0)}$. (4 pts)

   (3) Are X and Y independent if $\theta = 1$? (9 pts)

2. Suppose that the time to failure X has an exponential distribution with scale parameter λ. Let Y denote a random termination time and follow exponential distribution with scale parameter θ. Let T=min(X,Y), and define δ=1 if X ≤ Y, and δ = 0 if X > Y. Assume that X and Y are independent.

   (1) Find P(δ = 1) (5 pts)

   (2) Are T and δ independent? (10 pts)

3. Mr. Adams and Ms. Smith are betting on repeated flips of a coin. At the start of the game Mr. Adams has **a** dollars and Ms. Smith has **b** dollars, at each flip the loser pays the winner one dollar, and the game continues until either player is "ruined". Making use of the fact that in an equitable (公正) game each player's mathematical expectation is zero, find the probability that Mr. Adams will win Ms. Smith's **b** dollars before he loses his **a** dollars. **Please demonstrate the steps that you figure out this probability.** (15 pts)

4. Let $X_1, X_2, \ldots, X_n$ denote a random sample from uniform distribution with

$$f(x) = \begin{cases} \dfrac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

   Let $X_{(n)}$ be the largest order statistic, that is $Y = X_{(n)} = \max(X_1, X_2, \ldots, X_n)$.

   (1) Find the distribution (probability density function) of Y. (4 pts)

   (2) Find the limiting distribution (cumulative distribution function) of Y as $n \longrightarrow \infty$. (5 pts)

   (3) Find the limiting distribution of the appropriate standardized statistic $Z = n(\theta - Y)$ as $n \longrightarrow \infty$. (6pts)

   (4) Is Y a complete sufficient statistic for θ? (10 pts)

(背面仍有題目,請繼續作答)

5. Let (X_i Y_i) be independent pairs, i=1, 2, …, n, with X_i and Y_i are independent exponential random variables with $E(X_i)=\frac{1}{\theta\lambda}$ and $E(Y_i)=\frac{1}{\lambda}$. According to the common sense, one undergraduate student proposed estimating λ based on both X_i and Y_i, i=1, 2, …, n, simultaneously, and estimating θ based on X_i, i=1, 2, …, n, only. (30 pts)

(1) Do you agree his suggestion based on your intuition (直覺)? Why or Why not? (5 pts)

(2) Find maximum likelihood estimators (MLE) for θ and λ. (5 pts)

(3) Find minimum sufficient statistics for θ and λ. (5 pts)

(4) Find an unbiased estimator for θ.? (5 pts)

(5) Is the unbiased estimator in (4) a MVUE (minimum variance unbiased estimator) for θ? If yes, please find it; if no, please explain your reasons. (10 pts)