I. True or False (2 points each/Total 20 points) Write 0 for True and X for False.

1. The expectation of an exponential distribution is always positive.
2. The width of 95% confidence interval for the variance of a normal distribution does not depend on the sample size.
3. The sum of residuals of the fitted model $\hat{y} = \hat{\beta}x$ is always equal to zero.
4. The *sign* of β in linear model $y = \alpha + \beta x + \epsilon$ can be determined from the *sign* of the correlation coefficient of y and x.
5. We have $\Pr(X = x) = \Pr(X = n - x), x = 0, 1, \dots, n$, for $X \sim \text{Binomial}(n, p = 0.5)$.
6. For $X \sim \text{Normal}(\mu, \sigma^2)$, we have $\Pr(X \leq -1) = \Pr(X \geq 1)$ since the shape of normal distribution is symmetric.
7. A hypothesis being rejected at 5% significance level implies that it will also be rejected at 10% significance level.
8. For a normally distributed data (assuming the variance known), the width of 95% confidence interval for the mean is determined by the sample size.
9. For the fitted regression model $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$, we have $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = 0$.
10. The standard deviation is always smaller than the mean of a normal distribution.

II. Fill-in Blanks (2 points each blank/Total 40 points) Specify the number of each blank.

- For random sample $x_1, x_2, \dots, x_n$ from Poisson(λ), λ is the rate of occurrence, $\overline{x} = \sum_{i=1}^{n} x_i/n$,
  - $E(\overline{x}) = \boxed{1}$, $\text{Var}(\overline{x}) = \boxed{2}$;
  - for testing NH: $\lambda = 5$ versus AH: $\lambda \neq 5$, with large n, the distribution of test statistic $\boxed{3}$ is approximately normal under NH.
- For random sample $x_1, x_2, \dots, x_n$ from Normal(μ, σ^2), $\overline{x} = \sum_{i=1}^{n} x_i/n$ and $s^2 = \sum_{i=1}^{n}(x_i - \overline{x})^2/(n-1)$,
  - for testing NH: $\sigma^2 = 3$ versus AH: $\sigma^2 \neq 3$, the test statistic $\boxed{4}$ follows a χ^2_{n-1}-distribution under NH, and 95% confidence interval for σ^2 is $(\boxed{5}, \boxed{6})$; (Carefully in using the notation of the form $\chi^2_{df,1-\alpha}$ for the $100(1-\alpha)$th percentile of χ^2_{df}-distribution.)
  - $E(s^2) = \boxed{7}$ and $\text{Var}(s^2) = \boxed{8}$; (Noted that $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$, and the expectation and variance of a χ^2_{df}-distribution are df and $2 \times df$, respectively.)
  - applying Taylor's expansion, $s \approx \sigma + (s^2 - \sigma^2)/(2\sigma)$, we have $E(s) \approx \boxed{9}$ and $\text{Var}(s) \approx \boxed{10}$;

(背面仍有題目,請繼續作答)

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- the 100pth percentile $x_p = $ (11);
  (need to express in terms of μ, σ and z_p where $\Phi(z_p) = p$, $\Phi(\cdot)$ is the distribution function of standard normal.)
- and x_p can be estimated by $\hat{x}_p = $ (12), we have $E(\hat{x}_p) \approx$ (13) and $\mathrm{Var}(\hat{x}_p) \approx$ (14);
- with available data of size $n = 16$, $\overline{x} = 11$, $s^2 = 0.25$, then we can estimate the 95th percentile $x_{0.95}$ by $\hat{x}_{0.95} = $ (15), along with standard error $\sqrt{\mathrm{Var}(\hat{x}_{0.95})} \approx$ (16).
  (Noted that $z_{0.95} = 1.645$.)

• For random sample $x_1, x_2, \ldots, x_{25}$ from Normal$(\mu_x, 3^2)$, and, independently, $y_1, y_2, \ldots, y_9$ from Normal$(\mu_y, 2^2)$,

- $3\overline{x} - 2\overline{y} \sim$ Normal((17), (18));
- for testing NH: $\mu_x - \mu_y = 5$ versus AH: $\mu_x - \mu_y \neq 5$, the test statistic (19) follows (20) distribution under NH.

III. Derivation and Calculation (5 points each/Total 40 points):

• For random variable X, with $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, calculate $\Pr(|X - \mu| \leq 2\sigma)$ for

(a) Distribution is Exponential$(\theta = 4)$, where θ is the mean and the distribution function of Exponential(θ) is $F(x;\theta) = 1 - \exp(-x/\theta)$.

(b) Distribution is Normal$(17, 6^2)$.
(Noted that 2 is the 97.72th percentile of standard normal distribution.)

• For random sample $y_i \sim$ Binomial(m_i, p), $i = 1, 2, \ldots, n$,

(c) Derive the maximum likelihood estimator $\hat{p}_{\text{MLE}}$ of p.

(d) Calculate $\mathrm{Var}(\hat{p}_{\text{MLE}})$.

• For response variable $y_i \sim$ Exponential(θ_i), $\theta_i = E(y_i) = \beta x_i$, where x_i is explanatory variable, (i=1, 2, ..., n)

(e) Derive the (ordinary) least squares estimator $\hat{\beta}_{\text{LSE}}$ of β.

(f) Calculate $\mathrm{Var}(\hat{\beta}_{\text{LSE}})$.

(g) Derive the maximum likelihood estimator $\hat{\beta}_{\text{MLE}}$ of β.

(h) Calculate $\mathrm{Var}(\hat{\beta}_{\text{MLE}})$.

Noted that the density function of Exponential(θ) is $f(x;\theta) = \exp(-x/\theta)/\theta$.