1 Multiple Choice 3%× 8

For each question, there is one and only one appropriate answer.

1. Which of the following statements is/are **not** true?
   i. If $P(A\cap B\cap C) = P(A)P(B)P(C)$, then A, B, and C are independent events.
   ii. An empty set is independent from any other set.
   iii. For any nonempty subset A of the sample space S, $\{A, A^c\}$ is a partition of S.
   A. i B. ii C. iii D. i,ii E. i,iii F. ii,iii G. i,ii,iii

2. Let X and Y be the waiting time for the occurrence of the first event of two independent Poisson processes with equal rate 1. Which of the following statements is **not** true? Let $U = X + Y$ and $V = Y/X$.
   A. The conditional distribution of X given $U = u$, $X|U = u$ follows a uniform distribution.
   B. U follows a Gamma distribution.
   C. The ratio of X to U, X/U is uniformly distributed over $(0, 1)$.
   D. The conditional distribution of V given $X = x$, $V|X = x$ is uniformly distributed over $(0, 1)$.
   E. The range of V does not depend on the range of X.

3. Which of the following statements is/are **not** true?
   i. Suppose the conditional distribution of X given $Y = y$, $X|Y = y \sim N(y, \sigma^2)$, and $Y \sim N(\mu, \sigma^2)$, then the marginal distribution of X is $X \sim N(\mu, \sigma^2)$.
   ii. If a discrete random variable X is memoryless, then it has to be a geometric random variable.
   iii. The sum of two independent binomial random variable is a binomial random variable as well.
   A. i B. ii C. iii D. i,ii E. i,iii F. ii,iii G. i,ii,iii

4. What is the following statements is/are **not** true?
   i. For any nonempty subset A of the sample space S, $\{A, A^c, S, \emptyset\}$ is a σ-field.
   ii. For any sequence of sets $C_1, C_2, \ldots$, we have

$$P(\bigcap_{n=1}^{\infty} C_n) = \lim_{n\to\infty} P(C_n)$$

(背面仍有題目,請繼續作答)

iii. If $E \subset F$, then $P(E) \leq P(F)$ is one of the axiom of probability measure.
A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii

5. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with X_i uniformly distributed over interval $(0,1)$, and U_n be the maximum of the first n of the X_i. Which of the following statement(s) is/are **not** true?
i. $U_n \xrightarrow{P} 1$
ii. $n(1-U_n) \xrightarrow{P} 1$
iii. $n(1-U_n) \xrightarrow{d} \Gamma(1,1)$
A. i B. ii C. iii D. i,ii E. i,iii F. ii,iii G. i,ii,iii

6. Which of the following statement(s) is/are true?
i. The distribution function of a negative binomial distribution is a step function with jumps at positive integers and zero.
ii. Let $F_X(x)$ be a distribution function of random variable X, then $P(X = x) = 0$, where F_X is continuous at x.
iii. A distribution function has to be right-continuous.
A. i B. ii C. iii D. i,ii E. i,iii F. ii,iii G. i,ii,iii

7. Let X, Y be two random variables, which of the following statement(s) is/are true?
i. Given the marginal distributions of X, Y, the joint distribution of X, Y can be determined consequently.
ii. If X is independent from Y, then $h(X)$ is independent from $g(Y)$ for any function h, g.
iii. If X is independent from Y, then the joint range of X and Y has to be a rectangular.
A. i B. ii C. iii D. i,ii E. i,iii F. ii,iii G. i,ii,iii

8. Let $X_1, X_2, \ldots$ be a sequence of random variable with $\mathrm{E}(X_n) = \mu_n$ and $\mathrm{Var}(X_n) = \sigma_n^2$. Which of the following statement(s) is/are **not** true?
i. If $X_n \xrightarrow{P} a$, then we have $\mu_n \longrightarrow a$ and $\sigma_n^2 \longrightarrow 0$.
ii. If $X_n \xrightarrow{P} 5$ and $Y_n \xrightarrow{P} 2$, then $X_n \log(Y_n) \xrightarrow{P} 5\log(2)$
iii. If $\mathrm{Var}(X) = 0$, then $P(X = \mu) = 1$, where $\mathrm{E}(X) = \mu$.
A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii

2 Fill in the Blanks 3% × 8

1. Let $X_n \sim F_{n,1}$, what is your best approximated value of c such that $P(X_n > c) = 0.95$ when $n \to \infty$? ____A____

2. $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a n-dimensional random vector, and distributed as a multinomial distribution with parameters n, and $(p_1, p_2, \ldots, p_n)$. What is covariance between X_i and X_j? ____B____

3. Let $X_i, i = 1, \ldots, n$ are independently distributed as Poisson distribution with parameter $i\theta$ ($\mathbf{E}(X_i) = i\theta$). It can be shown that this model has a complete sufficient statistic, find the best unbiased estimator of θ^2 ____C____.

4. If X and Y are independent standard normal random variables, what is the distribution of Y/X, the ratio of Y to X? ____D____

5. A judge is 65% sure that a suspect has committed a crime. During the course of the trial a witness convinces the judge that there is 85% chance that the criminal is left-handed. If 23% of the population is left-handed and the suspect is also left-handed. How certain should the judge be of the guilt of the suspect? ____E____

6. X is uniformly distributed over the interval $(-1, 2)$, what is the density function of $Y = X^2$? ____F____

7. A random vector $\mathbf{X} = (X_1, X_2, X_3)$ have joint moment-generating function
$$M(t_1, t_2, t_3) = (1 - t_1 + 2t_2)^{-4}(1 - t_1 + 3t_3)^{-3}(1 - t_1)^{-2}.$$
Find the correlation coefficient between X_1 and X_2. ____G____

8. The covariance matrix of a random vector is given as
$$\begin{pmatrix} 4 & 2 & -3 \\ 2 & 10 & -5 \\ -3 & -5 & 16 \end{pmatrix}$$
and $Y_1 = X_1 - 2X_2 + 3X_3 - 4$, $Y_2 = 2X_1 + X_2 - 3X_3 + 5$. What is the value of the covariance between Y_1 and Y_2? ____H____

Problems

1. (14%) Suppose that you want to write a program to generate random numbers from a given continuous distribution, and what you have is the random number generator which can generate random number uniformly from the interval $(0, 1)$. What can you do? Explain and prove why you could do so.

2. (16%) Suppose that we observe $X_1, X_2, \ldots, X_n$, and X_i are independently uniformly distributed on the interval $(0, \theta)$. Find the best unbiased estimator of θ.

3. (12%) Prove and interpret

$$\mathrm{Var}(X) = \mathrm{Var}[\mathbf{E}(X|Y)] + \mathbf{E}[\mathrm{Var}(X|Y)]$$

4. (10%) A man invites his girl friend to a fine hotel for a Sunday brunch. They decide to meet in the lobby of the hotel between 11:30 and 12:00. If they arrive at uniformly random times during this period, what is the probability that they will meet within 10 minutes?