1 Multiple Choice 4%× 8

1. Consider an i.i.d. sample $X_1, X_2, \ldots, X_n$ from a population with finite variance, and the sample mean is $\bar{X}_n = \sum_{i=1}^n X_i/n$ and the sample variance is $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)/(n-1)$, which of the following statements is/are true?

   i. $\bar{X}_n$ and S_n^2 are uncorrelated under a normal population.

   ii. $\bar{X}_n$ and S_n^2 are uncorrelated under any population.

   iii. $\bar{X}_n$ and S_n^2 are asymptotically uncorrelated under any population.

   A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G.i,ii,iii H. None

2. Consider the same situation as in the previous question, which of the following statements is/are true?

   i. S_n^2 is an unbiased estimator of σ^2 under a normal population.

   ii. S_n^2 is an asymptotically unbiased estimator of σ^2 under any population.

   iii. S_n is an asymptotically unbiased estimator of σ under any population.

   A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G.i,ii,iii H. None

3. Which of the following statements is/are NOT true?

   i. Under a symmetric continuous population, both sample mean and sample median converge in probability to μ, the population mean.

   ii. If X is a discrete random variable and

   $$P(X \geq j + k | X \geq j) = P(X \geq k),$$

   then the distribution of X is a negative binomial one.

   iii. $\mathrm{E}X|Y$ is a function of X

   A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii H. None

4. Which of the following statements is/are true?

i. Two events A and B are independent if and only if $P(A|B) = P(A)$

ii. A probability measure is a non-negative function.

iii. If $A \subset B$, then $A < B$

A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii H. None

5. Which of the following statement(s) is/are true?

i. A complete sufficient statistics is minimal sufficient.

ii. For any model, there has to be a minimal sufficient statistic.

iii. The best unbiased estimator is unique for any model with a minimal sufficient statistic.

A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii H. None

6. Which of the following statement(s) is/are true?

i. To be a test statistic for testing $\boldsymbol{\theta} \in N$ (null hypothesis) against $\boldsymbol{\theta} \in A$ (alternative hypothesis), the distribution of $T(\mathbf{X})$ has to be independent from any unknown parameter under null hypothesis.

ii. The size of a test is the probability of a type II error.

iii. If there is a UMP size-α test for a problem, it has to be an unbiased test.

A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii H. None

7. Let X, Y be two random variables, which of the following statement(s) is/are NOT true?

   i. If X is nondegenerate and $\mathrm{E}X < \infty$, then $\mathrm{E}[\exp(X)] > \exp(\mathrm{E}X)$.

   ii. If X is independent from Y, then $h(X)$ is independent from $g(Y)$ for any function h, g.

   iii. If X and Y have the same moment generating function, then they must have the same distribution.

   A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii H. None

8. Which of the following statement(s) is/are NOT true?

   i. If $Z \sim N(0,1)$ and $W \sim \chi^2_k$, then

   $$t = \frac{Z}{W/k}$$

   has a t-distribution with k degrees of freedom.

   ii. If X has a Poisson distribution with mean λ, then $\mathrm{Var}(X)$ is λ as well.

   iii. The summation of independent Exponential random variables follows a Gamma distribution.

   A.i B.ii C.iii D.i,ii E.i,iii F. ii,iii G. i,ii,iii H. None

2 Fill in the Blanks 4% × 8

1. $X_1, X_2 \overset{iid}{\sim} N(0,1)$, what is $\mathrm{E}(X_1/X_2)$? ____A____?

2. $\bar{X}_n$ is the sample mean based on a sample of size n from a distribution with mean μ and variance σ^2, find the asymptotic distribution of $n^{1/2}(\bar{X}_n^2 - \mu^2)$ ____B____?

3. Let X and Y have joint density function $f(x,y) = y/10$, and

   $$S_{(X,Y)} = \{(x,y); (1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}.$$

   Find the conditional expectation Y given $X = 3$. ____C____.

(背面仍有題目,請繼續作答)

4. The same in the previous question, find the conditional variance of Y given $X = 1$.
_____D_____

5. Let X and Y have joint density function $f(x, y) = 1/6$, and

$$S_{(X,Y)} = \{(x, y); (0,0), (0,1), (1,0), (2,0), (1,1), (0,2)\}.$$

Find the joint moment generating function of X and Y. _____E_____?

6. Let Y be uniformly distributed on $(0, 1)$, and the distribution of X is a uniform one on $(0, y)$, what is the covariance between X and Y?_____F_____

7. Let X be distributed as a Gamma distribution with parameter a and b and $\mathrm{E}X = ab$, fine $\mathrm{E}(1/X)$._____G_____

8. Let $\mathbf{X} = (X_1, \ldots, X_6)$ follow a multinomial distribution with parameter n and $\mathbf{p} = (p_1, \ldots, p_6)$. Further, let $U = X_1 + X_2$, find the conditional distribution of U given X_6.
_____H_____

3 Problems

1. (12%) Let $X_1, \ldots, X_n$ be a independent random sample from a population with distribution $N(0, \theta^2)$. Find the *Uniformly Most Powerful Test* for testing $H_o : \theta = 1$ against $H_a : \theta > 1$. If the size is $\alpha = 0.05$ and $n = 30$, describe the critical region of this test.

2. (12%) Let $X_1, \ldots, \ldots$ be a sequence of random variables with X_i is uniformly distributed on $(0, 1)$, and U_n is the minimum of the first n of the X_i. Find the limiting distribution of $\exp(-nU_n)$.

3. (12%) Explain the meanings of sufficient statistic and minimal sufficient statistic.