※ 考生請注意：本試題 □可 ☑不可 使用計算機

一、Calculus

1. **(10 %)** **(a)** (5%) Given $f(x)=x^3$. Find an equation of the line that is tangent to the graph of f and parallel to the given line: $3x-y+1=0$.
   (b) (5%) Given $f(x)=|x-1|$. Find the derivatives from the left and from the right at $x=1$ (if they exist). Is f differentiable at $x=1$?

2 **(10%)** Find the area of the region bounded by the two curves: $y=8-x^2$, $y=x^2$.

3 **(10%)** **(a)** (4%) Find the exact form of the function f from the information given $f''(x)=x^2-x, f'(1)=0, f(1)=2$.

   (b) (3%) Calculate by a substitution $\int_0^1 \frac{x+3}{\sqrt{x+1}}dx$.

   (c) (3%) Calculate the derivative $\frac{d}{dx}\left(\int_x^{x^2}\frac{dt}{t}\right)$.

4 **(10%)** **(a)** (5%) Show that $f(x)=x^3+2x-5$ has an inverse and find $(f^{-1})'(7)$.

   (b) (5%) Find a formula for $(f^{-1})'(x)$ given that f is one-to-one and satisfies $f'(x)=\frac{1}{f(x)}$.

5 **(10%)** **(a)** (5%) Find all intervals on which the graph of the function $f(x)=\frac{x-1}{x+3}$ is concave upward.

   (b) (5%) Let $f''(x)=3x^2-4$ and $f(x)$ have critical numbers -2, 0 and 2. Use the Second Derivative Test to determine which critical numbers, if any, gives a relative maximum.

6 **(10%)** Find the Taylor series expansion of the natural logarithm function: ln(2+x) in x.

7 **(10%)** Find all the stationary (critical) points, saddle points, and the local extreme values of $f(x,y)=-x^4+\frac{8}{3}x^3+16xy-4y^2$, $-\infty<x<\infty, -\infty<y<\infty$,

(背面仍有題目,請繼續作答)

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二、Linear Algebra

1. Consider a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where (15%)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

and let $V = span(S)$.

(a) (5%)Find a set of vectors, say U, which is a basis for V.

(b) (5%)Find all possible $k \in \mathrm{R}$ such that

$$\vec{v}_k = \begin{bmatrix} k^2 \\ k \\ 1 \end{bmatrix} \in V.$$

(c) (5%)Find the subspace

$$W = \left\{\vec{w};\ \vec{u}^T\vec{w} = 0,\ \text{where } \vec{u} = a\vec{v}_1 + b\vec{v}_2 \text{ and } a, b \in \mathrm{R}\right\}$$

(Write your answer in the form $W = span\{\vec{w}_1, ..., \vec{w}_n\}$ where w_i's are linearly independent.)

2. Consider the equi-correlation matrix, for some $r \in (-1, 1)$, (15%)

$$\mathbf{A} = \begin{bmatrix} 1 & r & r & r \\ r & 1 & r & r \\ r & r & 1 & r \\ r & r & r & 1 \end{bmatrix}.$$

(a) (5%) Find $det(\mathbf{A})$.

(b) (10%) Do the Eigen-decomposition, *i.e.* $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T$ where $\boldsymbol{\Lambda}$ is a diagonal matrix with eigenvalues of $\mathbf{A}$ as its diagonal elements and $\mathbf{Q}$ consists of eigenvectors of $\mathbf{A}$. Note that columns of $\mathbf{Q}$ are orthonormal.