※ 考生請注意：本試題 □可 ☑不可 使用計算機

I. (7 points × 6 = 42 points) Let $X=(X_1,X_2)^T$ be a random vector distributed as trinomial (n,p_1,p_2), $0\le X_1+X_2\le n, X_i\ge 0, i=1,2, 0<p_1+p_2<1$, $p_i>0$, $i=1,2$.

1 For $n=1$, write down the probability mass function ($p.m.f.$) of X. Find the marginal $p.m.f.$ of X_i.

2. (Continued) Find $E(X_i), i=1,2,$ and $Cov(X_1,X_2)$

3. For general n, find the conditional distribution of $X_1 \mid X_2=x_2$.

4. (Continued) Find $E(X_1 \mid X_2=x_2)$ and verify that $E(X_1 \mid X_2=x_2)$ can be written in a linear form $\beta_0+\beta_1 x_2$.

5. (Continued) The β_0 and β_1 are functions of $E(X_i), Var(X_i), i=1,2,$ and $Cov(X_1,X_2)$. Verify that

$$\beta_0=g_0(E(X_i),Var(X_i),i=1,2,Cov(X_1,X_2))=E(X_1)-Corr(X_1,X_2)\sqrt{\frac{Var(X_1)}{Var(X_2)}}E(X_2)$$

and

$$\beta_1=g_1(E(X_i),Var(X_i),i=1,2,Cov(X_1,X_2))=Corr(X_1,X_2)\sqrt{\frac{Var(X_1)}{Var(X_2)}},$$

where $Corr(X_1,X_2)$ is the Pearson correlation coefficient of X_1 and X_2

6. Prove or disprove that $Var(X_1)=Var(E(X_1 \mid X_2))+E(Var(X_1 \mid X_2))$.

II. (8 points) Let X_1,X_2,X_3 be $i.i.d.$ sample from a gamma distribution with $p.d.f.$

$$f_X(x;r,\lambda)=\frac{\lambda^r x^{r-1}e^{-\lambda x}}{\Gamma(r)}, x>0,\ \gamma>0.$$

Find the distribution of $\overline{X}_3=(X_1+X_2+X_3)/3$

III. (7 points × 5 = 35 points) The R&D Department of ABC Group wants to establish a model for its petroleum industry about equipment VOCs leak emission rate so that a better sampling plan can be designed. It is known that the VOCs leak emission rate X for equipment LF follows a lognormal distribution, i.e.,

$Y=\ln X \sim N(\mu_Y,\sigma_Y^2)$. If we have a random sample $Y_1,Y_2,\ldots,Y_n$

1 Find $E(X)$ and $Var(X)$ (背面仍有題目,請繼續作答)

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2. Based on $Y_1, Y_2, ..., Y_n$, find the MLEs of μ_Y and σ_Y^2, and the MLE of $E(X) = g(\mu_Y, \sigma_Y^2)$

3. It is hoped the estimated relative error rate, defined as

$$\left| \frac{\hat{E}(X) - E(X)}{E(X)} = \frac{\hat{g}(\mu_Y, \sigma_Y^2) - g(\mu_Y, \sigma_Y^2)}{g(\mu_Y, \sigma_Y^2)} \right|,$$

should be less than τ with confidence level larger than $1-\alpha$. Let the MLE of (μ_Y, σ_Y^2) be $(\hat{\mu}_Y, \hat{\sigma}_Y^2)$

(1). Find the distribution of $\sqrt{n}(\hat{\mu}_Y - \mu_Y)$. Find the asymptotic distribution of

$$\sqrt{n}(\hat{\sigma}_Y^2 - \sigma_Y^2)$$

(2).It can be shown that

$$\hat{g}(\mu_Y, \sigma_Y^2) - g(\mu_Y, \sigma_Y^2) \approx \frac{\partial g(\mu_Y, \sigma_Y^2)}{\partial \hat{\mu}_Y}(\hat{\mu}_Y - \mu_Y) + \frac{\partial g(\mu_Y, \sigma_Y^2)}{\partial \hat{\sigma}_Y^2}(\hat{\sigma}_Y^2 - \sigma_Y^2)$$

Show that $\sqrt{n}(\hat{g}(\mu_Y, \sigma_Y^2) - g(\mu_Y, \sigma_Y^2))$ asymptotically follows

$$N\Big(0,\ (g(\mu_Y, \sigma_Y^2))^2 \sigma_Y^2 + \frac{1}{2}(g(\mu_Y, \sigma_Y^2))^2 \sigma_Y^4\Big),$$

where $\hat{g}(\mu_Y, \sigma_Y^2)$ is the MLE of $g(\mu_Y, \sigma_Y^2)$

(3). For given μ_Y, σ_Y^2 and $0 < \alpha < 1, \tau$, find the minimum value of n such that

$$P\Big(\Big|\frac{\hat{g}(\mu_Y, \sigma_Y^2) - g(\mu_Y, \sigma_Y^2)}{g(\mu_Y, \sigma_Y^2)}\Big| \le \tau\Big) \ge 1 - \alpha$$

IV. (15 points) Suppose we observe x and want to test

$H_0 : x$ is from f_0 vs. $H_1 : x$ is from f_1

where

x	1	2	3	4	5	6
$f_0(x)$	0.01	0.18	0.02	0.25	0.02	0.52
$f_1(x)$	0.10	0.10	0.08	0.22	0.20	0.30

1. (8 points) Find the most powerful (MP) test for $H_0 . f_0$ vs. $H_1 . f_1$ at α = 0.05. Find the power of the test.
2. (7 points) If we set α = 0.10, what is the MP test? What is the power of the test?