MIS - M.S. Entrance Exam (1998)

Calculus

1. (10%) Prove $|x+y| \le |x|+|y|$ for $\forall\, x, y \in R$

2. (10%) Evaluate $\lim_{x\to 0} \frac{|x|}{x}\cos x$ and $\lim_{x\to 0} \frac{|x|}{x}\cos x$. Does the limit as $x\to 0$ exist?

3. (10%) Find the y intercept of the tangent to the curve $y = x \sin x$ at $(\pi, 0)$.

4. (10%) If $f(x) = 2x - \sin x$, find a number c between 0 and π such that $f(\pi) - f(0) = f'(c)(\pi - 0)$. What theorem does this illustrate?

5. (10%) If f is continuous in the closed interval I with endpoints a and b, show that there exists a point $c \in I$ such that $\int_a^b f = f(c)(b-a)$.

6. (10%) Find an approximate value of $\ln 5$ as follows. (a) Express $\ln 5$ as an integral. (b) Compute a Riemann sum approximating the integral by dividing the interval of integration into four subintervals of equal length and evaluating the integrand at the midpoint of each subinterval.

7. (10%) Find the area of the region below the curve $y = 1/\sqrt{1-x^2}$, above the x axis, and between the lines $x = \pm 1$, or conclude that it does not exist.

8. (10%) Classify the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+1}}$ as absolutely convergent, conditionally convergent, or divergent.

9. (10%) Use a Maclaurin series to estimate $\sqrt{e}$.

10. (10%) Find the extreme values of $f(x, y, z) = x + y - z$ on the sphere $x^2 + y^2 + z^2 = 3$.