※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

註：每題配分 20 %，資料或條件不足時，請自行假設。

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{bmatrix}$

   (1) Prove that A is invertible.

   (2) Use Gauss-Jordan elimination to compute A^{-1}.

   (3) Let $b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, and solve the equation Ax=b.

2. Let $A = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{bmatrix}$

   (1) Write down the characteristic polynomial of A.

   (2) Find all eigenvalues of A and their corresponding eigenspaces.

   (3) Find an invertible real matrix S and a real diagonal matrix D such that $A = SDS^{-1}$.

3. Let $A = \begin{bmatrix} 0 & 0 & 0 & & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & & 0 & a_2 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & 0 & & -1 & a_{n-1} \end{bmatrix} \in M_{n\times n}(R)$

   Compute $\det(A + tI_n)$, where I_n is the n-by-n identity matrix, and t is an arbitrary real number.

4. Let $P_2(R) = \{a + bx + cx^2 | a, b, c \text{ are real numbers}\}$ be the vector space over real numbers with the ordinary addition and scalar product,

   (That is, $(f + g)(x) \coloneqq f(x) + g(x)$, and $(cf)(x) \coloneqq c(f(x))$, $\forall f(x), g(x) \in P_2(R), \forall c \in R$)

   $T: P_2(R) \to P_2(R)$ is a map defined by $T\big(p(x)\big) = \frac{d}{dx}[p(x)] = P'(x)$

   (1) Show that T is a linear transformation over real numbers.

   (2) Show that $\beta = \{1, x, x^2\}$ is a basis for $P_2(R)$.

   (3) Let $<,>$ be an inner product on $P_2(R)$ defined by $< f(x), g(x) >:= \int_0^1 f(t)g(t)dt$, apply Gram-Schmidt process to find an orthonormal basis γ from $\beta = \{1, x, x^2\}$.

5. Let $A = \begin{bmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{bmatrix}$ be a transition matrix

(1) Find the probability vector for A.

(2) Find $B = \lim_{m\to\infty} A^m$.

(3) Calculate e^B, where $e^X := \sum_{n=0}^{\infty} \frac{X^n}{n!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots$ whenever the series converge.