(1) We are familiar with the formula:

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}\text{, a polynomial has degree 2.}$$

Please derive a formula for the sum: $1^3+2^3+3^3+\ldots+n^3= f(n)$ in details.

(*Hint*: $f(n)$ is a polynomial of degree 4) (20%)

(2) Let $\mathbf{M}_{22}$ be the vector space of all 2×2 matrices, $\mathbf{P}_2$ be the vector space of all real polynomials with degree ≤ 2, and $\mathbf{R}^2$ be the 2-dimensional Euclidean vector space.

Define $S:\mathbf{M}_{22}\to\mathbf{P}_2$ by $S(\mathbf{A})=(3c-d)x^2+(b+2c)x+(a-c)$, where $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Define $T:\mathbf{P}_2\to\mathbf{R}^2$ by $T(a_0+a_1x+a_2x^2)=(a_0-a_1, 2a_1+a_2)$.

Please give the formula for $T\circ S:\mathbf{M}_{22}\to\mathbf{R}^2$. (20%)

(3) Show that $\mathbf{A}$ is diagonalizable by finding a matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{A}=\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$. (20%)

(4) Let X and Y be independent random variables each geometrically distributed with parameter p. Find $P(\min(X,Y)=X)=P(Y\geq X)$. (20%)

(5) Suppose n balls are distributed into n boxes so that all of the n^n possible arrangements are equally likely. Compute the probability that only box 1 is empty. (20%)