系所組別： 電信管理研究所乙、丙組
考試科目： 線性代數

※ 考生請注意：本試題 □可 ☑不可 使用計算機

(1) (20%) Let P_n denote the vector space of polynomials of degree $\le n$. Let $T: P_2 \to P_3$ be a linear transformation defined by $T\left(p(x)\right) = xp(x)$.
$B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are the bases of P_2 and P_3, respectively; where $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$, $\mathbf{u}_3 = x^2$; $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = 1 - x$, $\mathbf{v}_3 = x + x^2$, $\mathbf{v}_4 = x - x^3$

(a) Find the matrix for T such that $[T]_{B',B}[\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$

(b) Verify the matrix obtained in part (a) for every vector $\mathbf{x} = c_0 + c_1 x + c_2 x^2$ in P_2.

(2) (20%) Show that the functions $f_1 = 1$, $f_2 = e^x$, and $f_3 = e^{2x}$ form a linearly independent set of vectors in $C^2(-\infty, \infty)$ by using ***Wronskian***. $C^n(-\infty, \infty)$ denotes the subspace of n times differentiable continuous function on the interval $(-\infty, \infty)$.

(3) (20%) Let $W = span\{(1,2,3),(4,5,6)\}$ be a subspace in $\mathbf{R}^3$.

(a) Find a basis for $W^\perp$, where $W^\perp$ is the subspace orthogonal to W.

(b) Show that the vectors $(1,2,3)$, $(4,5,6)$ and the basis of $W^\perp$ from part (a) form a basis for $\mathbf{R}^3$.

(4) (20%) Let X and Y be continuous random variables having joint density function given by

$$f(x,y) = \begin{cases} n(n-1)(y-x)^{n-2}, & 0 \le x \le y \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

Please compute the conditional expectation of Y given $X = x$ and the expectation of Y.

(5) (20%) Let $X_1, \ldots, X_n$ be independent and identical distributed random variables having variance σ^2. Please compute $Cov\left(X_i - \bar{X}, \bar{X}\right)$.