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1. Use the Mean Value Theorem to show that:
$\frac{h}{1+h} < \ln(1+h) < h$ if either $h \in (-1,0)$ or $h > 0$ . (10%)

2 (a) Suppose that $f, g$ are integrable in [a,b]. Then prove the integral form of the Schwarz inequality $\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$, with equality if and only if f(x) = kg(x) in [a,b] for some constant k.

(b) For $f, g$ in (a), Prove the integral form of the Minkowski inequality:
$\sqrt{\int_a^b (f+g)^2} \le \sqrt{\int_a^b f^2} + \sqrt{\int_a^b g^2}$ . (15%)

3. Find the following limits :
(1) $\lim\limits_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$
(2) $\lim\limits_{k\to\infty}\sum_{n=1}^{k}\frac{1}{n(n+1)(n+2)}$ (10%)

4. Let $f_n(x) = (\sin nx)/n$, and for each fixed real x , let $f(x) = \lim\limits_{n\to\infty} f_n(x)$ .

Show that $\lim\limits_{n\to\infty} f_n'(0) \ne f'(0)$ . (10%)

5. (a) If 0<x<1, prove that $(1+x^n)^{1/n}$ approaches to a limit as $n \to \infty$ and compute this limit.
(b) Given a>0 , b>0 , compute $\lim\limits_{n\to\infty}(a^n+b^n)^{1/n}$ . (15%)

6. Let $g(x) = xe^{x^2}$ and let $f(x) = \int_1^x g(t)(t+\frac{1}{t})dt$ . Compute the limit of $f''(x)/g''(x)$ as $x \to +\infty$ . (10%)

7. (a) Show that $\int_0^\pi x f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$ .
(b) Use (a) to deduce $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \pi\int_0^1 \frac{1}{1+x^2}dx$ . (15%)

8. A truck is to be driven 300 miles on a freeway at constant speed of miles per hour. Speed laws require $30 \le x \le 60$ . Assume that fuel costs 30 cents per gallon and is consumed at the rate of $2 + x^2/600$ gallons per hour. If the driver's wages are D dollars per hour and if he obeys all speed laws, find the most economical speed and the cost of trip if (a) D=1 (b) D=2 . (15%)