(甲.乙組)

20% 1. Solve the ordinary differential equation

$$xy'' + y' - 4\frac{y}{x} = 1 \quad , \; 1 \le x \le 2$$

$$y(1) = 0, \quad y(2) = 1$$

20% 2. A conservative vector field is given by

$$\vec{F} = (y^2 \cos x + z^3)\,\vec{i} + (2y \sin x - 4)\,\vec{j} + (3xz^2 + 2)\,\vec{k}$$

(a) Find the corresponding potential function ϕ, ($\vec{F} = \nabla\phi$).

(b) Evaluate the line integral $\int_c \vec{F}\cdot d\vec{r}$, where the path of integration is $x = 2t+1$, $y = t^2$, $z = -1$, $0 \le t \le 1$.

20% 3. The area element and volume element in Cartesian coordinates (x, y, z) are $dA = dx\,dy$, $dV = dx\,dy\,dz$. A transform of coordinates is given by $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$.

(a) Derive the arc length ds in terms of du, dv, dw.

(b) Derive the expression for the area element in (u, v, w) coordinates.

(c) Derive the expression for the volume element in (u, v, w) coordinates.

20% 4. Denote the Laplace Transform of $f(t)$ as $\mathcal{L}\{f(t)\} = F(s)$.

(a) Show that $F'(s) = \mathcal{L}\{-t f(t)\}$.

(b) Show that $\int_s^\infty F(x)\,dx = \mathcal{L}\{t^{-1} f(t)\}$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = ?$

20% 5.

$$\nabla^2\phi = 0, \quad a \le r \le b$$

$$\nabla^2 \sim \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}$$

$$\phi(a, \theta) = 0, \quad \phi(b, \theta) = 1$$

(a) State whether the solution is independent of θ or not.

(b) Find ϕ.