1. (18%) Solve the following differential equations

(i) $2yy' + 2x + 3\cos y - 3xy'\sin y = 0$, $y(0) = \pi/2$,
(ii) $y'' + 2y' + y = \delta(t-1)$, $y(0) = 2$, $y'(0) = 3$.

2. (18%) (i) Discuss all possible solutions of the complex integral

$$\oint_C \frac{1}{(z-1-i)^n}\, dz,$$ where the contour C is $|z-1| = 2$ and n is an integer.

(ii) Evaluate the integral

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2}.$$

3. (12%) The standard form of Legendre differential equation is written as

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

Suppose $|x| < 1$, by letting $x = \cos\phi$, show that the equation can be transformed into

$$\frac{d^2y}{d\phi^2} + \cot\phi\,\frac{dy}{d\phi} + n(n+1)y = 0.$$

4. (16%) What kind of conic section is represented by the following quadratic form? Transform it to principal axes. Express $[x, y, z]^T$ in terms of the new coordinates.

$$5x^2 + 8xy + 5y^2 + 4xz + 4yz + 2z^2 = 100.$$

5. (18%) Verify the divergence theorem by working out $\int_V \nabla\cdot\mathbf{v}\; dv$ and $\int_S \mathbf{v}\cdot\mathbf{n}\; d\sigma$ and showing the results are equal, where

S: $x^2 + y^2 + z^2 = 1$; $z \geq 0$; $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

6. (18%) Consider the problem of steady–state temperature distribution in a rectangular plate. Suppose that the two edges $x = 0$ and $y = 0$ of a thin rectangular plate are maintained at zero temperature,

$$T(0,y) = 0,\ T(x,0) = 0,$$

and the two other edges $x = \ell$ and $y = \ell$ are maintained at a temperature distribution $T(x,\ell) = f(x)$ and $T(\ell,y) = g(y)$, respectively, until steady state conditions are realized. Find the temperature distribution throughout the plate.