1. (i) Evaluate $\int_C x\,dz$, where C consists of two line segments joining the points $1 \to 1+2i$, and $1+2i \to 2i$. (10%)

(ii) Evaluate $\oint_C (a^2-z^2)/(a^2z+z^3)\,dz$, where C is any simple closed curve enclosing the points $z = 0$ and $z = \pm ai$. (10%)

2. Consider the differential equation $[r(x)y']' + [p(x) + \lambda q(x)]y = 0$, where $r(x)$, $r'(x)$, $p(x)$ and $q(x)$ are continuous functions on some interval $a \le x \le b$ and λ is a real parameter. Suppose the end conditions are given by

$$a_1 y(a) - a_2 y'(a) = 0,\ b_1 y(b) - b_2 y'(b) = 0. \tag{1}$$

This is known as a **Sturm–Liouville** problem.

(i) If at least one coefficient in each equation in (1) is nonzero and $r(x)$ and $q(x)$ are positive on $a < x < b$, what are the basic properties of λ and $y(x)$? (8%)

(ii) Find the eigenvalues and eigenvectors of the problem (12%)

$$(xy')' + \lambda y/x = 0,\ y(1) = y(e) = 0.$$

3. (20%) Determine the solution $\phi = \phi(r, \theta)$ for the problem

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0,\ 0 \le r < a,\ 0 \le \theta \le 2\pi,$$

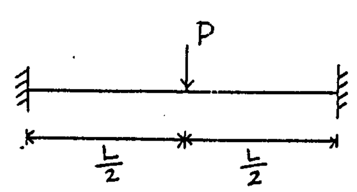
with boundary condition prescribed as

$$\phi(a, \theta) = \begin{cases} 1, & 0 < \theta < \pi, \\ 0, & \pi < \theta < 2\pi. \end{cases}$$

4. (20%) The deflection of an Euler–Bernoulli beam is governed by

$$EIy'''' = q(x),$$

where EI is the flextural rigidity and $q(x)$ is the external force function. Use the Laplace transform to obtain $y(x)$ for a fixed beam subject to a concentrated load P at the central point.



5. (10%) Given three symmetric matrices

$$L_1 = \begin{bmatrix} a_1 & b_1 \\ b_1 & -c_1 \end{bmatrix},\ L_2 = \begin{bmatrix} a_2 & b_2 \\ b_2 & -c_2 \end{bmatrix},\ L = \begin{bmatrix} a & b \\ b & -c \end{bmatrix},$$

in which they are connected by the relationship $L L_1^{-1} L_2 - L_2 L_1^{-1} L = 0$. Show, by direct expansion, this relation is equivalent to one scalar constraint:

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

6. (10%) Evaluate the integral $\oint_C (x^3+y^3)dx + (2y^3-x^3)dy$, where C is the unit circle.