(1) Evaluate $\int_c x\,dz$, where c consists of two line segments joining the points $1 \rightarrow 1+2i$, and $1+2i \rightarrow 2i$. (10%)

(2) The coordinates r, ϑ and their unit base vectors $\mathbf{e}_r, \mathbf{e}_\vartheta$ are shown in Fig. 1.

(a) Show that $\dfrac{d\mathbf{e}_r}{d\vartheta} = \mathbf{e}_\vartheta$ and $\dfrac{d\mathbf{e}_\vartheta}{d\vartheta} = -\mathbf{e}_r$ (5%)

(b) A point P moving in the x-y plane. The position vector $\mathbf{R}$ can be expressed by $\mathbf{R} = r\,\mathbf{e}_r$. Derive the expressions for the velocity and acceleration. (10%)

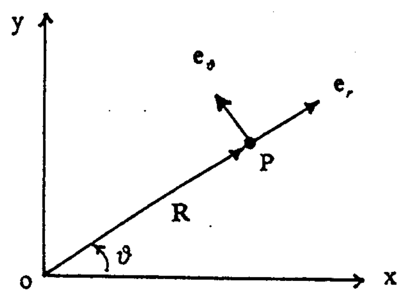


Fig. 1.

(3) Determine the solution $\phi = \phi(r, \vartheta)$ for the problem:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2} = 0 \qquad 0 \le r < a, 0 \le \vartheta \le 2\pi$$

with boundary condition prescribed as

$$\phi(a, \vartheta) = \begin{cases} 1 & 0 < \vartheta < \pi \\ 0 & \pi < \vartheta < 2\pi \end{cases}$$ (15%)

(4) A mass-spring system is shown in Fig. 2. The matrix form for free vibration of the system can be written as

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{0} \qquad (1)$$

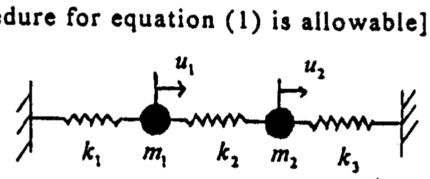
where $\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix}$, $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$; m_i (i = 1, 2) denotes the lumped mass and k_i (i=1, 2, 3) the stiffness coefficient; and $u_1(t), u_2(t)$ are the displacements.

(a) Let $m_1 = m_2 = 2$, $k_1 = 8$, $k_2 = 5$, $k_3 = 8$. Find a modal matrix $\mathbf{Q}$ which transforms the matrices $\mathbf{M}$ and $\mathbf{K}$ into diagonal matrices. (10%)

(b) Setting $\mathbf{u} = \mathbf{Q}\,\mathbf{x}$ in which $\mathbf{x} = (x_1, x_2)^T$ is the normal coordinates, and transforming the dependent variables in equation (1) from $\mathbf{u}$ to $\mathbf{x}$, one obtains two uncoupling differential equations for x_1 and x_2. Write down the solution process and solve the equations for x_1 and x_2, then find $\mathbf{u}$. (10%)

[No other solution procedure for equation (1) is allowable]

Fig. 2.



(5) The Dirac delta function $\delta(t)$ can be defined as $\delta(t)=\lim_{a\to 0}\dfrac{H(t)-H(t-a)}{a}$

where $H(t)$ denotes the step function.

(a) Find the Laplace transform of $\delta(t)$. (5%)

(b) The governing equation for the deflection of a Euler-Bernoulli beam is given by

$$EI\frac{d^4y(x)}{dx^4}=q(x)$$

where EI = the flexural rigidity, $q(x)$= the external force distribution function. Use the Laplace transform to obtain $y(x)$ for a fixed beam subject to a concentrated load at the central point. (see Fig. 3). (15%)

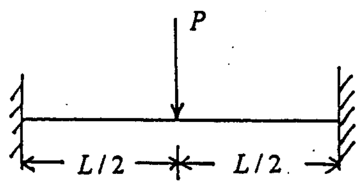


Fig. 3

(6) Consider the differential equation $[r(x)y']'+[p(x)+\lambda q(x)]y=0$, where $r(x), r'(x), p(x)$ and $q(x)$ are continuous functions in $a \le x \le b$ and λ is a real parameter. Suppose the end conditions are given by

$$a_1\,y(a)-a_2\,y'(a)=0, \qquad b_1\,y(b)-b_2\,y'(b)=0. \qquad (2)$$

This is known as a **Sturm-Liouville** problem.

(a) If at least one coefficient in each equation in (2) is nonzero, and $r(x)$ and $q(x)$ are positive on $a<x<b$, what are the basic properties of λ and $y(x)$? (10%)

(b) Find the eigenvalues and eigenvectors of the problem

$$(xy')'+\lambda y/x=0, \quad y(1)=y(e)=0. \qquad (10\%)$$