1. (18%) Solve the following differential equations:

(I). $\dfrac{dy}{dx}=\dfrac{x+y-5}{x-y+1}$,

(II). $x^2y''+xy'-y=\dfrac{1}{x}$.

2. (16%) Find the solution of the initial value problem

$$\frac{d^2y}{dt^2}+4y=f(t), \qquad y(0)=0, \qquad y'(0)=0$$

where $f(t)$ is defined as

$$f(t)=\begin{cases}1 & \text{for } 0\le t<\pi \\ -1 & \text{for } \pi\le t<2\pi\end{cases} \qquad \text{and} \qquad f(t+2\pi)=f(t).$$

3. (14%) Find the eigenvalues and associated eigenfunctions of the differential eguation

$$y''+2y'+(1+\omega^2)y=0$$

subject to the boundary conditions $y=0$ at $x=0$ and $x=\pi$. Verify that the eigenfuctions are orthogonal with the weighting function $w(x)=e^{2x}$.

4. (16%) Two identical simple pendula are connected by an elastic spring as shown in Fig. 1. For small oscillations the equations of motion are

$$ml\ddot{\theta}_1=-mg\theta_1-kl(\theta_1-\theta_2),$$
$$ml\ddot{\theta}_2=-mg\theta_2+kl(\theta_1-\theta_2).$$

Find the natural frequencies and normal modes of vibration, and find the general solution of the system.

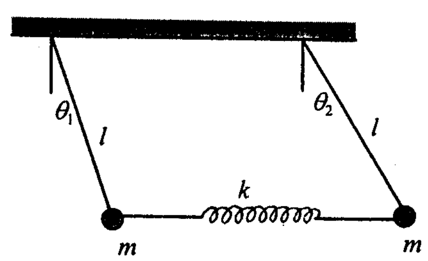

Fig.1.

(背面仍有題目,請繼續作答)

5. (18%) Use residues to evaluate the following integrals

(I). $\int_{-\infty}^{\infty} \frac{x\, sin ax}{x^4+4} dx,$ $(a>0)$,

(II). $\int_0^{\pi} \frac{cos 2\theta}{1-2a cos\theta + a^2} d\theta,$ $(-1<a<1)$.

6. (18%) Heat conduction in a bar with insulated ends is modeled by the boundary value problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0<x<L,\ t>0),$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \qquad (t>0),$$
$$u(x,0) = f(x) \qquad (0<x<L).$$

where $u(x, t)$ is the temperature distribution, a is a constant.

(I). Let $u(x, t) = X(x)T(t)$ to solve this problem.

(II). Find the solution for $f(x) = cos\left(\frac{2\pi x}{L}\right)$.