1.(20%)

For a given basis in a three-dimensional space

$g_1 = [1,-1,2]$, $g_2 = [0,1,1]$, $g_3 = [-1,-2,1]$.

(i) Find the coordinate of the vector $h = [3,3,6]$ in this basis.

(ii) Let us introduce a set of reciprocal base vectors g^1, g^2, g^3 so that

$g_i \cdot g^i = 1, \quad i = 1,2,3; \quad g_i \cdot g^j = 0, \quad i \neq j.$ Find the reciprocal base vectors g^1, g^2, g^3.

(iii) Find the coordinate of the vector $h = [3,3,6]$ in the reciprocal basis.

2.(20%)

(i) Describe in detail the divergence theorem and Green's theorem.

(ii) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = 4xy\mathbf{i} - 8y\mathbf{j} + 2\mathbf{k}$ and C is along the ellipse $x^2 + 4y^2 = 4$, $z = 0$ counterclockwise from $(0,-1,0)$ to $(0,1,0)$.

3.(20%) Solve the set of differential equations

$$2\frac{dx}{dt} - 3x + y = 4e^t,$$

$$x + 2\frac{dy}{dt} - 3y = 0.$$

4.(20%) Evaluate the integral using the theory of residues

$$\int_{-\infty}^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx$$

5.(10%) Consider the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} - z = 0, \; z(x,0) = x/2 + e^x.$$

(i) Let $u = \frac{x+y}{\sqrt{2}}, v = \frac{-x+y}{\sqrt{2}}$ and derive the equation in terms of the variables u, v.

(ii) Solve the differential equation and write the solutions in terms of x, y.

6.(10%) Find the Fourier series of the function of period 2π, $f(x) = |x|, |x| < \pi$.