1. Let $f(\theta)$ be a continuous function of period 2π with Fourier series

$$f(\theta) = \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

(i) (10%) Find the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the disk $x^2 + y^2 < 1$ with boundary values f.

(ii) (10%) Suppose $f(\theta) = \cos^2\theta$, find the Fourier coefficients a_n, b_n.

2. (20%) In the x_1-x_2-x_3 space, suppose there is a body $\mathcal{D}$ depicted by the inequality

$$\mathcal{D}: \quad 1 - (x_1^2 + x_2^2 + x_3^2 + 2x_1x_2x_3) > 0,$$

and x_1, x_2 and x_3 are in the ranges:

$$|x_1| < 1, \; |x_2| < 1, \; |x_3| < 1,$$

Describe the geometry of the body $\mathcal{D}$. (Hint: let $x_3 = a$, where $|a| < 1$, and plot the section)

3. (i) (10%) Compute

$$\int_{|z|=2} z e^{1/z} dz.$$

(ii) (10%) Evaluate the integral

$$\frac{1}{2\pi} \int_{|z|=1} \frac{\omega(z)\bar{\omega}(1/z)}{z-\xi} dz,$$

where $\omega(z) = \sum_{n=0}^{\infty} a_n z^n$ and ξ is a complex number inside the unit circle of the complex plane, namely $|\xi| < 1$.

(背面仍有題目,請繼續作答)

4. Suppose $\varphi(x, y)$ satisfies the Laplace equation in a certain simply connected region Ω

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0, \quad \text{in } \Omega$$

with the boundary condition on the boundary C

$$\frac{d\varphi}{dn} = yn_1 - xn_2, \quad \text{on } C,$$

where n is the exterior normal to the boundary C of the region Ω.

(i) (10%) Evaluate the integral

$$\int_C \frac{d\varphi}{dn} ds = ?$$

where s denotes the arc length of C.

(ii) (5%) By letting

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x},$$

show that the governing equation and the boundary condition change to

$$\nabla^2\psi = 0, \quad \text{in } \Omega; \quad \psi = (x^2 + y^2)/2 + \text{constant.} \quad \text{on } C$$

Introduce the function Ψ defined by

$$\Psi = \psi - (x^2 + y^2)/2.$$

(iii) (10%) Show that the function Ψ satisfies

$$\nabla^2\Psi = -2, \quad \text{in } \Omega; \quad \Psi = \text{constant.} \quad \text{on } C$$

5. (15%) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^2.$$