1. (20%) Solve the boundary value problem by separation of variables:

$$\nabla^2\varphi = 0,$$

with the boundary conditions

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=a_1} = \left.\frac{\partial\varphi}{\partial x}\right|_{x=a_2} = 0, \quad \left.\frac{\partial\varphi}{\partial y}\right|_{y=b} = \left.\frac{\partial\varphi}{\partial y}\right|_{y=-b} = x.$$

2. (i) (10%) Find an analytic function $f(z) = u + \mathrm{j}v$, $z = x + \mathrm{j}y$, for which

$$u = xy\left(x^2 - y^2\right).$$

(ii) (10%) Find the value of the integral

$$\oint_C \frac{dz}{z^3(z+4)},$$

where C is the circle $|z| = 2$.

3. (15%) Solve

$$y'' - 4y' + 13y = \delta(t-1), \quad y(0) = 0,\ y'(0) = 3.$$

4. (15%) Determine the general solution

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= -2, \\ -2x_1 + 3x_2 - x_3 + 2x_4 &= 5, \\ 4x_1 - 2x_2 + 2x_3 - 3x_4 &= 6. \end{aligned}$$

5. (15%) Compute the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma$, where $\mathbf{F}(x,y,z) = x^2\mathbf{i} + 2y\mathbf{j} + 4z^2\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 \le 4, 0 \le z \le 2$.

6. Legendre's differential equation is given by

$$\left(1 - x^2\right)y'' - 2xy' + n(n+1)y = 0.$$

One of the solution is a polynomial of degree n, known as Legendre's polynomial $P_n(x)$,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n, \quad n = 0, 1, 2, \cdots.$$

It is known that Legendre's polynomials are orthogonal in the interval of $(-1, 1)$, namely

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} being the Kronecker's delta, defined as

$$\delta_{mn} = \begin{cases} 1, & \textit{when } m = n, \\ 0, & \textit{when } m \ne n. \end{cases}$$

(i) (5%) Write down the expressions for $P_0(x), P_1(x), P_2(x)$ and $P_3(x)$.

(ii) (10%) Given a function $f(x) = x^3$, we wish to expand it as a series of Legendre's polynomial

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x).$$

Find the coefficients of a_0, a_1, a_2 and a_3.

718003 A2