1. Solve the following differential equations

a) $(1+x)^2 y'' - 2y = 0$ (12 %)

b) $y' + y = f(x)$

where $f(x) = 1$ when $0 \le x \le 1$ and

$f(x) = 0$ when $x > 1$

and $y(0) = 0$. (12%)

2. Evaluate the following integrals

a) $\int_{|z|=1} \frac{\bar{z}}{z} dz$ (10%)

b) $\int_{|z|=1} \frac{dz}{\sqrt{6z^2 - 5z + 1}}$ (10%)

where the square root is the positive $\sqrt{2}$ at $z = 1$.

3.

a) Identify and characterize the singularities of

(i) $f(z) = z / \sin z$ (4%)

(ii) $f(z) = \sin z / z$ (4%)

b) Locate and classify the singular points of the following differential equations

(i) $(x-1)y'' + \sqrt{x}y = 0$ $(x \ge 0)$ (4%)

(ii) $xy'' + y \sin x = 0$ (4%)

4. Solve the Laplace equation

$\nabla^2 u = 0$

in the annular region defined by $a < |z| < b$ with $u = 1$ at $r = b$

and $\frac{\partial u}{\partial r} = 0$ at $r = a$. (15%)

5. Laguerre's equation is

$$xy'' + (1-x)y' + ny = 0$$

with solutions defined as $y(x,n) = L_n(x)$.

a) Show that solutions $L_k(x)$, $L_j(x)$, j,k integers, are orthogonal with respect to the weighting function e^{-x} over the interval $[0,\infty)$, i.e.,

$$\int_0^\infty e^{-x} L_k(x) L_j(x)\, dx = 0 \quad \text{if} \quad j \neq k. \qquad (10\%)$$

b) What boundary conditions must the $L_n(x)$ satisfy in the deriving above results? (5%)

6. Use Stokes's theorem to determine the value of the integral

$$\oint_c [(1+y)z\, dx + (1+z)x\, dy + (1+x)y\, dz]$$

where c is a closed curve in the plane x-2y+z=1. (10%)