1. (20%) Define $\mathbf{R}$ is a rotation matrix with a 90 degree clockwise given as

$$\mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Suppose that $\phi(x,y)$ is a scalar function and $\mathbf{q}(x,y)$ is a divergence-free vector (2×1) field, show that (i) $\mathbf{R}\nabla\phi$ is divergence-free, i.e., $\nabla\cdot\mathbf{R}\nabla\phi = 0$ (ii) $\mathbf{Rq}$ is curl-free, i.e. $\nabla\times\mathbf{Rq} = 0$.

2. (20%) Let $f_1(\theta)$ and $f_2(\theta)$ denote the normal derivatives of the harmonic function $u(r,\theta)$ on the boundaries $|z| = r_1$ and $|z| = r_2$ of the annulus $r_1 \le |z| \le r_2$. Show that (i) $f_1(\theta)$ and $f_2(\theta)$ must satisfy the condition

$$r_1\int_{-\pi}^{\pi} f_1(\theta)d\theta + r_2\int_{-\pi}^{\pi} f_2(\theta)d\theta = 0.$$

(ii) find the field solution $u(r,\theta)$.

3. (10%) Given four complex numbers a_3, a_1, a_{-1} and a_{-3} in which $|a_3| = |a_1| = |a_{-1}| = |a_{-3}| = 1$. Suppose now that $a_3\bar{a}_1\bar{a}_{-1}a_{-3} \in \mathrm{Re}$ (real number), $a_3\bar{a}_1\bar{a}_1a_{-1} \in \mathrm{Re}$, and $a_1\bar{a}_{-1}\bar{a}_{-1}a_{-3} \in \mathrm{Re}$, where the over bar denotes the complex conjugation. Find all possible solutions of a_3, a_1, a_{-1} and a_{-3} that satisfy the conditions. (Hint: define $\xi_k = \arg a_k$) .

4. (10%) Show that if $u(z)$ is harmonic in $|z - z_0| < R$, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{i\theta}\right)d\theta, \quad 0 < r < R.$$

5 (20%) Solve the system of differential equations

$$\begin{cases} 2x' - y' + x + y = t \\ x' + y' + 4x = 3. \end{cases}$$

6 (20%) Let

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{6} \\ \sqrt{6} & 3 \end{bmatrix},$$

and $\mathbf{B} = \mathbf{A}^5 - 3\mathbf{A}^4 - 2\mathbf{I}$.

(i) Determine the eigenvalues and corresponding eigenvectors of $\mathbf{B}$.

(ii) Determine whether $\mathbf{B}$ is positive definite.

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