1. Find the particular solution of the following equation

$$(x+1)\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=e^x(x+1)^2. \qquad (20)$$

2. Solve the following differential equation

$$\frac{d\mathbf{x}}{dt}=\begin{pmatrix}5 & 8\\ -6 & -9\end{pmatrix}\mathbf{x}+\begin{pmatrix}1\\ t\end{pmatrix}$$

where

$$\mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}, \qquad (20)$$

3. The temperature distribution in a homogeneous spherical solid filling the closed region $x^2+y^2+z^2\le 1$ at time t is given by $u=(z^2-z)e^{-2t}$. Let $\vec{n}$ be the unit outer normal on the boundary of the sphere. Find the point at which $\partial u/\partial n$ is minimum. (15)

4. Given an analytic function $f(z)=F_1(x,y)+iF_2(x,y)$, where $z=x+iy$ and $i=\sqrt{-1}$. If the real part $F_1(x,y)$ and the imaginary part $F_2(x,y)$ of $f(z)$ serve as the components of a vector $\vec{F}$, i.e.

$$\vec{F}=F_1\vec{i}+F_2\vec{j}, \qquad (10)$$

where $\vec{i}$ and $\vec{j}$ denote the unit vector in x- and y-direction respectively. Then, is the vector $\vec{F}$ a conservative one ? why ?

5. Given a velocity field as

$$\vec{v}=y\vec{i}-z\vec{j}+yz\vec{k}$$

find the surface integral

$$I=\int\int_S \vec{v}\bullet\vec{n}dA, \qquad (15)$$

where $\vec{n}$ is a unit normal vector in the outer direction of the surface

$$S: x=\sqrt{y^2+z^2}; \qquad y^2+z^2\le 1$$

6. The displacement $u(x,t)$ of a semi-infinite string is governed by the following partial differential equation

$$c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}, \quad 0<x, \quad 0<t, \qquad (20)$$

where c is a constant. With the initial conditions

$$u(x,0)=0, \quad \frac{\partial u(x,0)}{\partial t}=0,$$

and the excitation $f(t)$ at one end of string, that is,

$$u(0,t)=f(t)$$

then, what is the solution of $u(x,t)$?