1. (20%) Solve the boundary value problem

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(\mathbf{x})=-2,\ \mathbf{x}\in\Omega;\quad \Phi(\mathbf{x})=0,\ \mathbf{x}\in\partial\Omega$$

where Ω is a circle centered at the point $\mathbf{x}_0$ with radius R, i.e.

$$\Omega:|\mathbf{x}-\mathbf{x}_0|\le R,\quad \partial\Omega:|\mathbf{x}-\mathbf{x}_0|=R.$$

Express $\Phi(\mathbf{x})$ in terms of $\mathbf{x},\ \mathbf{x}_0$ and R.

2. (i) (10%) Evaluate

$$\oint_C \frac{z^2}{2z-1}dz,\ C:|z|=1.$$

(ii) (10%) Given $z_1=1+i$, $z_2=1+i\sqrt{3}$, $z_3=\sqrt{3}-i$. Find

$$\arg\left(\frac{z_1 z_2}{z_3}\right).$$

3. (i) (10%) Setting

$$x=e^z,$$

show that

$$x\frac{d}{dx}=\frac{d}{dz},\quad x^2\frac{d^2}{dx^2}=\frac{d^2}{dz^2}-\frac{d}{dz}.$$

(ii) (10%) Solve the differential equation

$$x\frac{dy}{dx}+y+x^2y^2=0.$$

Hint: introduce a new variable $v=y^{-1}$.

(背面仍有題目,請繼續作答)

4. (i) (10%) Solve the algebraic set of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 1 \\ 3x_1 - 3x_2 - 5x_3 &= 1. \end{aligned}$$

(ii) (10%) Suppose the matrices A and B possess the same set of linearly independent eigenvectors so that they can be diagonalized through

$$S^{-1}AS = \Lambda_A, \quad S^{-1}BS = \Lambda_B,$$

where Λ_A and Λ_B are diagonal matrices. Show that in this case $AB = BA$.

5. (i) (10%) Evaluate the line integral

$$\int_C \mathbf{f} \cdot \mathbf{dx}$$

where $\mathbf{f}(\mathbf{x}) = 5z\,\mathbf{i} + xy\,\mathbf{j} + x^2z\,\mathbf{k}$ and C is the straight line from $(0,0,0)$ to $(1,1,1)$.
(ii) (10%) Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,d\sigma$$

where $\mathbf{F} = 2x\,\mathbf{i} + 3y\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the hemisphere $x^2+y^2+z^2 = 9$, $z \geq 0$.