1. (i) (10%) Expand $f(z)$

$$f(z) = \frac{1}{(z+1)(z+3)}$$

in a Laurent series valid for $1 < |z| < 3$.

(ii) (10%) Evaluate the integral

$$\oint_C \bar{z}dz$$

where C is the circle $|z| = 2$.

2. (i) (10%) For a curve $x = t^2 + 1,\ y = 4t - 3,\ z = 2t^2 - 6t$, determine the unit tangent vector at the point where $t = 2$.

(ii) (10%) Evaluate

$$\iint_S \mathbf{x} \cdot \mathbf{n} dS$$

where $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{n}$ is the outward unit normal to S, and S is the surface of the sphere

$$(x-1)^2 + (y+3)^2 + z^2 = 4.$$

3. (10%) (i) Given the 3×3 matrix $\mathbf{P}$ of the form

$$\mathbf{P} = \begin{pmatrix} 7 & -2 & -4 \\ 3 & 0 & -2 \\ 6 & -2 & -3 \end{pmatrix}.$$

Find a matrix $\mathbf{C}$ such that $\mathbf{C}^{-1}\mathbf{PC}$ becomes a diagonal matrix.

(ii) (10%) Given a 3×3 symmetric matrix $\mathbf{A}$ defined by

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix},$$

what are the constraint conditions for the elements a_{ij} to ensure that the matrix $\mathbf{A}$ is positive definite?

(背面仍有題目,請繼續作答)

4. (20%) Determine the steady-state temperature $T(r, \theta)$ at points of the sector $0 \leq \theta \leq \alpha$, $0 \leq r \leq a$ of a circular plate if the temperature is maintained at zero along the straight edges and at a prescribed distribution $T(a, \theta) = T_0 =$ constant when $0 < \theta < \alpha$, along the curved edge. (Hint: solve $\nabla^2 T$ by separation of variables).

5. (i) (10%) Solve

$$\frac{dy}{dt} + 4y = 3H(t-2)\,e^{-t};\ \ y(0) = 0,$$

where $H(x)$ is the unit step function.

(ii) (10%) Solve

$$x^2 y'' - 3xy' + 3y = \ln x;\ \ y(1) = 1,\ \ y'(1) = 2.$$