本試題是否可以使用計算機：□可使用，☑不可使用　（請命題老師勾選）

1. (i) (10%) What is Bessel's equation of order n? Write down the solutions for $n =$ integer and $n \neq$ integer. (ii) (10%) What is Legendre's equation? Describe what you know about Legendre polynomials.

2. (20%) We consider the relation

$$x = r\cos\theta,\ y = r\sin\theta.$$

Then

$$r = \left(x^2 + y^2\right)^{\frac{1}{2}},\ \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Obviously, we can see that

$$\frac{\partial x}{\partial r} = \cos\theta. \tag{1}$$

(i) derive

$$\frac{\partial r}{\partial x}$$

in terms of r and θ, and compare with Eq. (1), (ii) show that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

3. (20%) Let Ω be a simply connected region in the xy-plane bounded by a piecewise smooth curve $\partial\Omega$. Let $\mathbf{T}$ denote the unit tangent vector to $\partial\Omega$ and $\mathbf{n}$ be the unit normal vector to $\partial\Omega$. Given a vector $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ in the plane, (i) describe the physical (or mathematical) meanings of the two line integrals

$$\oint_{\partial\Omega} \mathbf{F}\cdot\mathbf{T}ds, \quad \text{and} \quad \oint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}ds,$$

where s denotes the parameter of arc length, (ii) transform these two line integrals into double integrals in the plane.

(背面仍有題目,請繼續作答)

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4. (i) (10%) Find the solution of

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}, \quad \text{for } a > |b|.$$

(ii) (10%) Given $z = x + iy$, evaluate the integral

$$\int_C y dz$$

where C is the straight line joining $z = 1$ to $z = i$.

5. Given the second-order linear partial differential equation of two independent variables with constant coefficients

$$a\frac{\partial^2 w}{\partial x^2} + 2b\frac{\partial^2 w}{\partial x \partial y} + c\frac{\partial^2 w}{\partial y^2} + nw = f(x, y), \tag{2}$$

where a, b, c and n are constants. Using the substitution

$$\begin{aligned} u &= x\cos\alpha + y\sin\alpha, \\ v &= -x\sin\alpha + y\cos\alpha, \end{aligned}$$

and transform Eq. (2) into the form

$$A\frac{\partial^2 w}{\partial u^2} + 2B\frac{\partial^2 w}{\partial u \partial v} + C\frac{\partial^2 w}{\partial v^2} + nw = f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha).$$

(i) (15%) Find the expressions of A, B and C in terms of a, b, c and α, (ii) (5%) Under what condition, the partial differential equation (2) is referred to as an elliptic type.