系所組別：工程科學系甲組

考試科目：線性代數與機率　　考試日期：0223，節次：3

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。**請依題號順序作答。**

1. (a) Let $\mathbf{A}$ and $\mathbf{B}$ be $n\times n$ matrices over the filed $F^{n\times n}$. Prove that if $\mathbf{I}-\mathbf{AB}$ is invertible, then $\mathbf{I}-\mathbf{BA}$ is also invertible and $(\mathbf{I}-\mathbf{BA})^{-1}=\mathbf{I}+\mathbf{B}(\mathbf{I}-\mathbf{AB})^{-1}\mathbf{A}$. (10 分)

   (b) Prove that, if $\mathbf{A}$ is invertible, then $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues. (10 分)

2. Let $\mathbf{A}=\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}$, prove that the power of $\mathbf{A}$, i.e. $\mathbf{A}^m=\begin{bmatrix}\cos(m\theta) & \sin(m\theta)\\ -\sin(m\theta) & \cos(m\theta)\end{bmatrix}$, where m is an integer. (**Hint**: Use the method of induction.) (20 分)

3. Assume that 70% of an inventory of diodes comes from vendor 1 and the remaining 30% from vendor 2, and that 98% of the units from vendor 1 and 95% of those from vendor 2 give satisfactory performance. If we pick one diode randomly, then (a) What is the probability of selecting a unit that is made by vendor 1 and defective? (10 分) (b) What is the probability of selecting one that is defective, irrespective of vendor? (10 分)

4. If the random variable X is normal distributed, find the probability density function of
   (a) $Y=2X+1$. Besides, find the mean and variance of Y. (15 分)
   (b) $Y=2X^2+1$. (5 分)

5. There are two independent chi-square random variables, X_1 and X_2, with two degrees of freedom. The probability density function of a chi-square random variable X with two degrees of freedom is

$$f(x)=\frac{1}{2}e^{-\frac{x}{2}}.$$

   Please find the probability of the event

$$E=\left\{(x_1,x_2)\middle|\begin{array}{c}0\le x_1\le 1\\ 0\le x_1+x_2\le 3\end{array}\right\}.$$

   You can express your answer by a function of $e\doteq 2.71828$. (20 分)