※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (15%) Consider two vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in R^3$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in R^3$.

    (1) (4%) Find $\|\vec{u}\|$ and $\|\vec{v}\|$.

    (2) (5%) Find $proj_{\vec{v}}^{\vec{u}}$ (orthogonal projection of $\vec{u}$ on $\vec{v}$).

    (3) (6%) Find a vector $\vec{w} \in R^3$ such that $\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

2. (15%) Let $T_1, T_2, T_3, T_4 : R^2 \to R^2$ be the following linear transformations.

    T_1 : Projection on the *y-axis*.

    T_2 : Reflection with respect to the line L_1 of angle $45°$.

    T_3 : Rotation anticlockwise by the angle $30°$.

    T_4 : Contraction with factor $k = 0.5$.

    (1) (5%) Find the standard matrices A_1 A_2, A_3 and A_4 of $T_i(\vec{x}) = A_i\vec{x}, i = 1, \cdots, 4$, respectively.

    (2) (5%) Find the standard matrix of the composition $T_1 \circ T_2$.

    (3) (5%) Describe the composition $T_2 \circ T_3$ geometrically. Will the composition $T_2 \circ T_3$ be a rotation transformation? How angle will rotate?

3. (20%) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

    (1) (6%) Find the eigenvalues of A and its corresponding eigenvector.

    (2) (6%) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

    (3) (8%) Find A^{50} and exp^A.

4. (10%) Suppose that there are two websites, A and B, for buying smartphones. The site A receives 40% of all orders. Among the orders placed on site A, 85% arrive on time. Among the orders placed on site B, 95% arrive on time. Given that an order arrived on time, find the probability that the order was placed on site B.

5. (10%) Let X and Y be independent random variables, each of which has a standard normal probability density function (pdf). If $Z = Y - X + 3$, then find the mean and variance of Z.

6. (15%) Let X be a random variable with the probability density function (pdf)

$$f_X(x) = \begin{cases} k|x| & \text{if } -2 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(1) (5%) Find the value for k.

(2) (5%) Find $P(X \ge 1 \mid X \ge 0)$.

(3) (5%) Find $P(X \ge \mu)$, where $\mu = E[X]$.

7. (15%) Let the continuous random variables X, Y have joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{2x} & \text{if } 0 < y < x < 2, \\ \\ 0 & \text{otherwise.} \end{cases}$$

(1) (5%) Find $E(X)$ and $E(Y)$.

(2) (5%) Find the conditional probability density function (pdf) of Y given $X = x$, for all $0 < x < 2$.

(3) (5%) Find $E(Y \mid X = x)$ for all $0 < x < 2$.