國立成功大學

113學年度碩士班招生考試試題

編　號：109

系　所：工程科學系

科　目：線性代數

日　期：0202

節　次：第 3 節

備　註：不可使用計算機

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10 %) Determine whether the following statements are true (T) or false (F)? (A reasoning is required.)
   (1) (2%) If a system of linear equations has two different solutions, it must have infinitely many solutions.
   (2) (2%) Let $A = \begin{bmatrix} 2 & 6 & 40 \\ 98153 & -105 & 101 \\ 2 & 1 & 7 \end{bmatrix}$, then cofactor $C_{21} = -2$ and $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 1021$.
   (3) (2%) If $\|u\| = 1$, $\|v\| = \sqrt{2}$, and $u \cdot v = 1$, then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\frac{\pi}{3}$ radians.
   (4) (2%) Let the vector space V have two basis by $B = \{\sin x, \cos x\}$ and $B' = \{\sin x - \cos x, 3\cos x\}$, then the transition matrix from B to B' is $\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$.
   (5) (2%) If the inner product on P_2 is defined by $\langle f, g\rangle = \int_{-2}^{2} f(x)g(x)dx$, then $\langle 2+x, 1-x+x^2\rangle = \frac{2}{3}$.

2. (12%) Consider the following system of linear equations:

$$\begin{cases} x + y + z = a \\ 2x + y + 3z = b \\ 3x + 4y + 2z = c \end{cases} \quad \text{where } a, b, c \text{ are constants}$$

   (1) (4%) Determine the a, b, c such that the system has no solution.
   (2) (4%) Determine the a, b, c such that the system has a unique solution.
   (3) (4%) Determine the a, b, c such that the system has infinite solutions.

3. (14%) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, please
   (1) (7%) find the LU factorization of the matrix A.
   (2) (7%) find the QR factorization of the matrix B.

4. (20%) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ -2 & 0 & 3 & -3 \end{bmatrix}$.

    (1) (4%) Find the rank of A.

    (2) (4%) Find the nullity of A.

    (3) (4%) Find the nullity of A^T.

    (4) (4%) Find the basis for the column space of A.

    (5) (4%) Find the basis for the null space of A.

5. (14%) Let $T: R^3 \to R^4$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\-1\\-2\\0\end{bmatrix},\ T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\-3\\1\end{bmatrix},\ T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\-8\\2\end{bmatrix}.$$

    (1) (8%) Find $T\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}2\\0\\3\end{bmatrix}\right)$.

    (2) (6%) Is T one-to-one? Justify your answer.

6. (14%) For the linear operator $L: R^3 \to R^3, L(\mathbf{x}) = \begin{bmatrix} -x_1 + x_3 \\ -2x_2 \\ x_1 + 2x_3 \end{bmatrix}$, please find

    (1) (7%) $\ker(L)$

    (2) (7%) $L(S)$ for $S = span\{e_1, e_2\}$, where $e_1 = [1 \quad 0 \quad 0]^T$ and $e_2 = [1 \quad 0 \quad 0]^T$.

7. (16%) Consider the matrix $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -3 \end{bmatrix}$

    (1) (4%) Find the eigenvalues of A, and its corresponding eigenvectors.

    (2) (4%) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

    (3) (4%) Find the unique solution of the differential equation $\frac{dX(t)}{dt} = AX(t), t \geq 0$ with the initial condition $X(0) = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

    (4) (4%) What is the behavior of the above differential equation? Will it be converged? or diverged?