國立成功大學

114學年度碩士班招生考試試題

編　號：86

系　所：工程科學系

科　目：線性代數

日　期：0211

節　次：第 3 節

注　意：1.不可使用計算機
2.請於答案卷(卡)作答，於試題上作答，不予計分。

1. (14 %) Determine whether the following statements are true (T) or false (F)? (A reasoning is required.)

(1) (2%) If $\mathbf{det}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = -3$, then $det\left(\begin{bmatrix} i & h & g \\ f & b & a \\ c & e & d \end{bmatrix}\right) = 3$.

(2) (2%) If we have the polynomial $f(x) = det\left(\begin{bmatrix} x & 1 & x & 3 \\ x & 1 & 1 & 2x \\ 1 & 2 & x & 3 \\ 2 & -3x & 2 & 2x \end{bmatrix}\right)$, then the coefficient of the term x^4 is 24.

(3) (2%) If A and B are two $n \times n$ matrices with $trace(A) = -2$ and $trace(B) = 3$, then $trace(4A - 2B + 6B^T - 3A^T) = 12$.

(4) (2%) The transformation of the orthogonal projection on the x-axis in R^2 is one-to-one transformation.

(5) (2%) Given a 4×4 matrix A. If we have $\mathbf{b} = (1) \cdot \mathbf{a}_1 + (-3) \cdot \mathbf{a}_2 + (2) \cdot \mathbf{a}_3 + (-2) \cdot \mathbf{a}_4$ where $\mathbf{a}_i$ denotes the i-th column vector of A, then $\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ -2 \\ 2 \end{bmatrix}$ is the solution of equation $A\mathbf{x} = \mathbf{b}$.

(6) (2%) Let M_{22} denote the vector space consisting of all 2×2 matrices. If $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, we define $\langle A, B \rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$. Now, if we have $A = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}$, are A and B linearly independent?

(7) (2%) If A is a 4×3 matrix of rank 1, then the dimensions of $N(A^T)$ is 3.

2. (10%) Given a matrix $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$, please find (1) (2%) the row space of A; (2) (2%) the column space of A; (3) (2%) the null space of A; (4) (2%) the $rank(A)$; (5) (2%) the $nullity(A)$.

3. (15%) Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. Please (1) (5%) use the Gram–Schmidt process to find an orthonormal basis for the column space of A; (2) (5%) factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular; (3) (5%) solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

4. (20%) Suppose that A is a 3×3 matrix with eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = 1$, and corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} (\lambda_1 = 0), \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} (\lambda_2 = -1), \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} (\lambda_3 = 1).$$

(1) (5%) Find the matrix A.

(2) (5%) Find A^{10}.

(3) (5%) Find the unique solution of the differential equation $\frac{dY(t)}{dt} = AY(t), t \geq 0$ with the initial condition $Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(4) (5%) Find $Y(\infty) = ?$

5. (16%) Let $V = R_{\geq 0}$ denote the set of non-negative real numbers. Define the operation of addition, denoted by $\oplus$, $\mathbf{x} \oplus \mathbf{y} = \mathbf{x} + \mathbf{y}$ (the regular addition of $\mathbf{x}$ and $\mathbf{y}$) for all $\mathbf{x}, \mathbf{y} \in V$. Define the operation of scalar multiplication, denoted by $\otimes$, $\alpha \otimes \mathbf{x} = \mathbf{x}^{\alpha}$ for each $\mathbf{x} \in V$ and for any $\alpha \in R$. Thus, for this system, the addition of 2.5 and 4.3 is given by:

$$2.5 \oplus 4.3 = 2.5 + 4.3 = 6.8,$$

and the operation of scalar multiplication of -2 and $\frac{1}{10}$ is given by:

$$-2 \otimes \frac{1}{10} = \left(\frac{1}{10}\right)^{-2} = 100.$$

Is $(V, \oplus, \otimes)$ a vector space? Prove your answer.

6. (10%) Given two vectors $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, please find the vectors $\mathbf{p}$ and $\mathbf{z}$ such that $\mathbf{x} = \mathbf{p} + \mathbf{z}$ with $\mathbf{p} \parallel \mathbf{y}$ and $\mathbf{z} \perp \mathbf{p}$.

7. (15%) Let P_3 denote the vector space of polynomials of degree less than 3. And, the operator $L: P_3 \to P_3$ is defined by $L(f(x)) = x^2 f(1) + f'(x)$. ($where\ f'(x) \triangleq \frac{df(x)}{dx}$)

(1) (5%) Find the matrix A representing L with respect to the basis $\{x^2, x, 1\}$.

(2) (5%) Find the matrix B representing L with respect to the basis $\{x^2 + 1, 2x + 1, 1\}$.

(3) (5%) Find the matrix S such that $B = S^{-1}AS$.