(甲、乙組)

1. Solve

(10%)

$$\begin{aligned} x + 2y + 4z - w &= 3 \\ 3x + 4y + 5z - w &= 7 \\ x + 3y + 5z + 5w &= 4 \end{aligned}$$

2. Define a function f(t)=k on the set of rational number Q

(10%)

(a) Whether f is periodic or not? prove.

(b) Is f(t) + sin t periodic? explain.

(c) If you have proved f is periodic, then (I) what is the period of f?
(II) Does the fundamental period of f exist? explain.

3. Solve $y' + 2ay + (a^2 + b^2)\int_0^t y\,dt = f(t)$, $y(0)=k$

(10%)

4. If $x_1, \dots, x_n$ are eigenvectors corresponding, respectively, to the distinct eigenvalues $\lambda_1, \dots, \lambda_n$ of a hermitian matrix $H \in \mathbb{C}^{n\times n}$

(15%)

(a) prove that $\{x_1, \dots, x_n\}$ is a base.

In (b) and (c), the matrix inversion is not allowed to use.

(b) express any given vector x in terms of the combination of $x_1, \dots, x_n$.

(c) solve y in Hy=x.

5. Theorem: If f is a periodic function which satisfies the Dirichlet conditions and has nonzero jumps $F_1, F_2, \dots, F_m$ at the respective points $t_1 < t_2 < \dots < t_m$ in one period $d \le t \le d+2p$ of f, where t_1 may be d, but $t_m \ne d+2p$, then

(15%)

$$a_n = -\frac{p}{n\pi} b_n^{(1)} - \frac{1}{n\pi}\sum_{k=1}^{m} F_k \sin\frac{n\pi t_k}{p}, \qquad n \ne 0$$

$$b_n = \frac{p}{n\pi} a_n^{(1)} + \frac{1}{n\pi}\sum_{k=1}^{m} F_k \cos\frac{n\pi t_k}{p}$$

where $a_n^{(1)}$ and $b_n^{(1)}$ are coefficients of $\cos(n\pi t/p)$ and $\sin(n\pi t/p)$ in the Fourier expansion of f(t).

Question: Use the above theorem to find a_0, a_i, b_i in

$$f(t)=a_0 + \sum_{i=1}\left(a_i \cos\frac{i\pi t}{p} + b_i \sin\frac{i\pi t}{p}\right)$$

where $f(t)=t^3$, $-\pi \le t < \pi$

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6. Wave equation:

(15%)
$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

boundary conditions: $u(0,t)=u(1,t)=0$,
initial conditions: $u(x,0)=\sin(\pi x)$, $\left.\frac{\partial u}{\partial t}\right|_{(x,0)}=0$,

Find $u(0.5,1)$ by the method of Laplace Transforms.

7. Given $f(z)=\dfrac{(z^2+1)^3}{(z^2+2z+2)^2}$, solve $\oint_c \dfrac{f'(z)}{f(z)}dz$

(12%)

where c is the circle of $|z|=4$.

8. Transform $\iint_\Omega e^{(x-y)/(x+y)}\,dxdy$, where Ω is the region in the first quadrant between the lines $x+y=2$ and $x+y=3$, by making the transformation $u=x-y$ and $v=x+y$. Then evaluate the integral.
(13%)